Stability of Time Delay Systems Using Numerical Computation of Argument Principles

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Abstract: This paper proposes a new numerical method to check the stability of a general class of time delay systems. The proposed method checks whether there are characteristic roots whose real values are nonnegative through two steps. Firstly, rectangular bounds of characteristic roots whose real values are nonnegative are computed. Secondly, the existence of roots inside the bounds are checked using the numerical computation of argument principles. An adaptive discretization is proposed for the numerical computation of argument principles.

Keywords: Time delay systems, numerical method, argument principle.

1. INTRODUCTION

Consider the following time delay system:

$$\dot{x} = \sum_{i=0}^{N_1} A_i x(t-h_i) + \sum_{i=1}^{N_2} \int_{-h_{N_1+i}}^{0} R_i e^{P_i r} Q_i x(t+r)dr, \qquad (1)$$

where $x \in R^n$ is a state, $h_0 = 0$, and (A_i, P_i, Q_i, R_i) are real matrices of compatible sizes.

The system (1) encompasses a large number of distributed delay systems. For example, the closed-loop system in [1] is given in the form of (1). Another example can be found in [2], where the closed-loop system is of the form (1).

The characteristic equation of (1) is given by

$$\begin{aligned} f(s) = \det(sI - \textstyle\sum_{i=0}^{N_1} A_i e^{-sh_i} \\ -\textstyle\sum_{i=1}^{N_2} \int_{-h_{N_1+i}}^{0} R_i e^{P_i r} Q_i e^{sr} dr) = 0. \end{aligned} \qquad (2)$$

Although much attention has been paid to the stability of (1), a computable analytic method to check the stability of (1) is not completely known. For general time delay systems, there is a necessary and sufficient stability condition based on a Lyapunov functional [3]; however, the condition is given in the form of an operator Lyapunov equation, and numerical approximations are needed to compute it. For commensurate point delay systems $(h_i = hi,\ N_2 = 0)$, there is a stability condition [4] using the fact that the root of (2) varies continuously with respect to the change of h. This stability condition provides an interval $[h_l, h_u]$ inside which the stability of (1) does not change. Thus, if the system is stable for $h = h_l$, the system is stable for all $h \in [h_l, h_u]$. This stability condition is useful only for $h_l = 0$ since the condition itself does not provide a way to check the stability for $h_l \neq 0$. To overcome this problem, a method, which investigates the movement of roots of (2) on the boundary $h = h_u$, was devised [5][6].

For a general class of time delay systems, a currently available method [7] to check the stability of (1) is to compute closed-right half plane roots of (1). However, the stability check using the method is not an automatic process. Furthermore, it is not necessary to compute roots of (1) just to check the stability of (1); it suffices to check whether there are roots in the closed-right half plane. The nyquist theorem can be used for this purpose; however, the nyquist theorem is a graphical method and not a numerically amenable method. Hence a new method to check whether there are roots in the closed-right half plane is proposed in this paper. The basic idea is based on the fact that roots can only exist in a confined location of the closed-right half plane if they exist. Hence it suffices to check the confined location for stability instead of the whole closed-right half plane.

The paper is organized as follows. In Section II, the rectangular bounds of the closed-right half plane roots of (2) are derived; the roots of (2) in the closed-right half plane can only exist inside the rectangular bounds. In Section III, the existence of roots in the rectangular bounds is checked using the numerical computation of argument principles. In Section IV, several examples are given to illustrate the proposed method to check the stability of (1). The conclusion is given in Section V.

Notation

$\|A\|$ $2-$norm of matrix A

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$\mu(A)$ matrix measure of A[8]
$\lambda_i(A)$ the i-th eigenvalue of A
$\alpha(A)$ max Re $\lambda_i(A)$
$\mu\Delta(A)$ structured singular value of A with respect to Δ[9]
$\arg f(s)$ argument of complex function $f(s)$
$\operatorname{Arg} f(s)$ principal argument of $f(s)$

2. RECTANGULAR BOUNDS OF ROOTS WHOSE REAL VALUES ARE NONNEGATIVE

Since $\left|e^{-s}\right| \le 1$, $\operatorname{Re} s \ge 0$, the possible location ofroots of (2) whose real values are nonnegative is confined. In this section, two rectangular bounds of the location are provided in Theorem 1 and 2. The bounds of Theorem 1 are the generalization of the bounds of [10]; the bounds of Theorem 1 are easy to compute but rather conservative. The bounds of Theorem 2 are computed using linear matrix inequalities [11] and are less conservative than those of Theorem 1. When (A_i, Q_i, R_i) are small, it suffices to use the bounds of Theorem 1; when (A_i, Q_i, R_i) are large, it is desirable to use the bounds of Theorem 2 to avoid overly conservative bounds. Since (A_i, P_i, Q_i, R_i) are real, the roots of (2) are symmetric with respect to the real axis. Hence the bounds are also symmetric with respect to the real axis, and the rectangular bounds can be determined by two positive variables l_1 and l_2 (every closed-right half plane root s satisfies $\operatorname{Re} s \le l_1$ and $\left|\operatorname{Im} s\right| \le l_2$) as seen in The first bounds are given in the next theorem.

Theorem 1 : Let l_1 and l_2 be defined by

$$\begin{aligned} l_1 &:= \mu(A_0) + \sum_{i=1}^{N_1} \|A_i\| + \sum_{i=1}^{N_2} \gamma_i \\ l_2 &:= \mu(-jA_0) + \sum_{i=1}^{N_1} \|A_i\| + \sum_{i=1}^{N_2} \gamma_i, \end{aligned} \tag{3}$$

where γ_i is defined in Lemma 1. Then

$$\operatorname{Re} s \le l_1, \quad \left|\operatorname{Im} s\right| \le l_1, \quad \forall s \in W$$

where W is the set of roots of (2) whose real values are nonnegative.

Proof: Noting the properties of matrix measure μ [8],

$$\begin{aligned} \operatorname{Re} \lambda(A) &\le \mu(A), \\ \operatorname{Im} \lambda(A) &\le \mu(-jA), \\ \mu(A+B) &\le \mu(A) + \mu(B), \\ \mu(A) &\le \|A\|, \end{aligned}$$

we obtain

$$\begin{aligned} \operatorname{Re} s \le\ & \operatorname{Re} \lambda\Big(A_0 + \sum_{i=1}^{N_1} A_i e^{-sh_i} \\ &+ \sum_{i=1}^{N_2} \int_{-h_{N_1+i}}^{0} R_i e^{P_i r} Q_i e^{sr} dr\Big) \\ \le\ & \mu(A_0) + \sum_{i=1}^{N_1} \|A_i\| \\ &+ \sum_{i=1}^{N_2} \left\| \int_{-h_{N_1+i}}^{0} R_i e^{P_i r} Q_i e^{sr} dr \right\|, \\ \operatorname{Im} s \le\ & \operatorname{Im} \lambda\Big(A_0 + \sum_{i=1}^{N_1} A_i e^{-sh_i} \\ &+ \sum_{i=1}^{N_2} \int_{-h_{N_1+i}}^{0} R_i e^{P_i r} Q_i e^{sr} dr\Big) \\ \le\ & \mu(-jA_0) + \sum_{i=1}^{N_1} \|A_i\| \\ &+ \sum_{i=1}^{N_2} \left\| \int_{-h_{N_1+i}}^{0} R_i e^{P_i r} Q_i e^{sr} dr \right\|. \end{aligned}$$

Invoking Lemma 1, we obtain (3). □

The next lemma is concerned with the norm bounds of the distributed delay terms of (2).

Lemma 1: For $\operatorname{Re} s \ge 0$, the following is satisfied:

$$\left\| \int_{-h_{N_1+i}}^{0} R_i e^{P_i r} Q_i e^{sr} dr \right\| \le \gamma_i, \tag{4}$$

where γ_i is defined by

$$\gamma_i = \sum_{j=1}^{m_i-1} \Big[e^{\alpha(-P_i)(h_{N_1+i})} \sum_{k=0}^{j} (-1)^k (h_{N_1+i})^{j-k} \frac{1}{\alpha(-P_i)^k (j-k)!} - \frac{(-1)^j}{\alpha(-P_i)^j} \Big] \frac{\|R_i\| \|M_i\|^j \|Q_i\|}{\alpha(-P_i)}.$$

N_i comes from Schur decomposition of P_i:

$$U_i' P_i U_i = D_i + M_i,$$

where U_i is a unitary matrix, D_i is a diagonal matrix, M_i is a strictly upper triangular matrix, and m_i is the index of nilpotency of M_i (i.e., $m_i = \min(s : M_i^s = 0)$).

Proof:

$$\left\| \int_{-h_{N_1+i}}^{0} R_i e^{P_i r} Q_i e^{sr} dr \right\| \le \int_{0}^{h_{N_1+i}} \|R_i\| \left\|e^{-P_i r}\right\| \|Q_i\| dr$$

Using the following inequality [12]:

$$\left\|e^{P_i r}\right\| \le e^{\alpha(P_i) r} \sum_{j=0}^{m_i-1} \|M_i\|^j \frac{r^j}{j!},$$

we obtain

$$\left\| \int_{-h_{N_1+i}}^{0} R_i e^{P_i r} Q_i e^{sr} dr \right\|$$
$$\le \int_0^{h_{N_1+i}} e^{\alpha(-P_i)r} \sum_{j=0}^{m_i-1} \tfrac{r^j}{j!} \|R_i\| \|M_i\|^j \|Q_i\| dr$$
$$= \sum_{j=0}^{m_i-1} \int_0^{h_{N_1+i}} e^{\alpha(-P_i)r} r^j dr \tfrac{1}{j!} \|R_i\| \|M_i\|^j \|Q_i\|$$
$$= \sum_{j=0}^{m_i-1} \|R_i\| \|M_i\|^j \|Q_i\|$$
$$\tfrac{1}{j!}\left[\tfrac{e^{\alpha(-P_i)r}}{\alpha(-P_i)r} \left(\sum_{k=0}^{j} (-1)^k r^{j-k} \tfrac{j!}{\alpha(-P_i)^k (j-k)!} \right) \right]_0^{h_{N_1+i}}$$
$$= \sum_{j=0}^{m_i-1} \left[e^{\alpha(-P_i)(h_{N_1+i})} \sum_{k=0}^{j} (-1)^k (h_{N_1+i})^{(j-k)} \right.$$
$$\left. \tfrac{1}{\alpha(-P_i)^k (j-k)!} - \tfrac{(-1)^j}{\alpha(-P_i)^j} \right] \tfrac{\|R_i\| \|M_i\|^j \|Q_i\|}{\alpha(-P_i)},$$

where the following is used in the third equality:

$$\int_{t_2}^{t_1} x^k e^{ar} dr = \left[\frac{e^{ar}}{a} \left(\sum_{i=0}^{k} (-1)^i r^{k-1} \frac{k!}{a^k (k-i)!} \right) \right]_{t_2}^{t_1}.$$

□

The second bounds are given in the next theorem.

Theorem 2: If there exist $l_1 \in R > 0$, $X = X'$, $\in R^{n\times n} > 0$ $D = D_i' \in R^{n\times n} > 0$ and $d_i \in R > 0$ such that

$$\begin{bmatrix} E_1 & XA_1 & \cdots & XA_{N_1} & \gamma_1 X & \cdots & \gamma_{N_2} X \\ A_1' X & -D_1 & & & & & \\ \vdots & & \ddots & & & & \\ A_{N_1}' X & & & -D_{N_1} & & & \\ \gamma_{11} X & & & & -d_1 I & & \\ \vdots & & & & & \ddots & \\ \gamma_{N_2} X & & & & & & -d_{N_2} I \end{bmatrix} < 0, \quad (5)$$

where

$$E_1 = (A_0 - l_1 I)' X + X(A_0 - l_1 I) + \sum_{i=1}^{N_1} D_i + \sum_{i=1}^{N_2} d_i I,$$

then

$$\text{Re } s < l_1, \ \forall s \in W,$$

where W is the set of roots of 2 whose real values are nonnegative.

If there exist $l_2 \in R > 0$, $Y = Y' \in C^{n\times n} > 0$, $D_i = D_i' \in R^{n\times n} > 0$, and $d_i \in R > 0$ such that

$$\begin{bmatrix} E_2 & YA_1 & \cdots & YA_{N_1} & \gamma_1 Y & \cdots & \gamma_{N_2} Y \\ A_1' Y & -D_1 & & & & & \\ \vdots & & \ddots & & & & \\ A_{N_1}' Y & & & -D_{N_1} & & & \\ \gamma_{11} Y & & & & -d_1 I & & \\ \vdots & & & & & \ddots & \\ \gamma_{N_2} Y & & & & & & -d_{N_2} I \end{bmatrix} < 0, \quad (6)$$

where

$$E_2 = (-jA_0 - l_2 I)' Y + Y(-jA_0 - l_2 I) + \sum_{i=1}^{N_1} D_i + \sum_{i=1}^{N_2} d_i I, \text{then}$$

$$|\text{Im} s| < l_2, \ \forall s \in W.$$

Proof: (l_1 part) For notational simplicity, it is assumed that $N_1 = N_2 = 1$. It suffices to show that $f(s) \neq 0$, $\forall \text{Re} s \ge l_1$, which is equivalent to $f(s + l_1) \neq 0$, $\forall \text{Re} s \ge l_1$, $f(s + l_1)$ is given by

$$f(s + l_1) = \det(sI + l_1 I - A_0)$$
$$\det(I - (sI + l_1 I - A_0)^{-1}$$
$$(A_1 e^{-(s+l_1)h_1} + \int_{-h_2}^{0} R_1 e^{P_1 r} Q_1 e^{(s+l_1)r} dr)).$$

Note $\det(sI + l_1 I - A_0) \neq 0$, $\forall \text{Re} s \ge 0$ since $E_1 < 0$. It is easy to show that $f(s + l_1) \neq 0$, $\forall \text{Re} s \ge 0$ if

$$\det(I - (sI + l_1 I - A_0)^{-1} [A_1 \quad \gamma_1 I] \begin{bmatrix} \Delta_1 I \\ \Delta_2 \end{bmatrix}) \neq 0,$$

for all $\text{Re} s \ge 0$, $|\Delta_1| \le 1$, and $\|\Delta_2\| \le 1$. The condition (7) is a standard structured singular value problem [9]; (7) is satisfied if

$$\mu\Delta(\begin{bmatrix} I \\ I \end{bmatrix} (sI + l_1 - A_0)^{-1} [A_1 \quad \gamma_1 I]) < 1, \forall \text{Re} s \ge 0,$$

where

$$\Delta = \begin{bmatrix} \Delta_1 I & \\ & \Delta_2 \end{bmatrix}.$$

Note for any $D_1 = D_1' > 0$ and $d_1 > 0$, we have

$$\mu\Delta(\begin{bmatrix} I \\ I \end{bmatrix} (sI + l_1 - A_0)^{-1} [A_1 \quad \gamma_1 I])$$
$$\le \left\| \begin{bmatrix} D_1^{\frac{1}{2}} \\ d_1^{\frac{1}{2}} \end{bmatrix} (sI + l_1 - A_0)^{-1} \left[A_1 D_1^{-\frac{1}{2}} \quad \gamma_1 d_1^{-\frac{1}{2}} \right] \right\|,$$
$$\forall \text{Re} s \ge 0.$$

It is standard that (5) is equivalent to the right-hand side of the above inequality. Hence we obtain l_1. The l_2 part can be proved in the similar manner. □

3. NUMERICAL COMPUTATION OF ARGUMENT PRINCIPLES

Since the possible location of roots whose real values are nonnegative is confined, the system is stable if there is no root in the possible location. The existence of roots is checked using argument principles along the rectangular bounds given by Theorem 1 and 2. In this section, only usual stability is considered since α-stability is a direct extension of usual stability.

Argument principles provide the number of roots inside the contour of Fig. 1 (b) as seen in the next theorem. Recall that roots are symmetric with respect to the real axis; thus only the upper part is checked.

Theorem 3 : [13] Let Γ be a rectangular counterclockwise contour (see Fig. 1 (b)). Let (2) have no zeros on Γ. Then

$$\frac{1}{2\pi}\int_{\Gamma}\arg f(s)ds = L, \tag{8}$$

where L is the number of roots of $f(s)$ counting multiplicities interior to Γ.

For the purpose of a stability check, the integral along the real axis (Γ_4 in Fig. 1 (b)) can be omitted due to the next theorem.

Theorem 4 : Let (2) have no zeros on Γ_1, Γ_2, and Γ_3. Then

$$\frac{1}{2\pi}\left[\int_{\Gamma_1+\Gamma_2+\Gamma_3}\arg f(s)ds\right] = L_1 + \frac{1}{2}L_2, \tag{9}$$

where L_1 is the number of roots of $f(s)$ counting multiplicities interior to Γ, and L_2 is the number of roots of $f(s)$ counting multiplicities on Γ_4.

Proof: If there is no root on Γ_4 (i.e. $L_2 = 0$), then the proof is obvious from Theorem 3 since $\int_{\Gamma_4}\arg f(s)ds = 0$. If $L_2 \neq 0$ (for simplicity, it is assumed $L_2 = 1$), suppose a small circle ($b \to c \to d \to c' \to b$ in Fig. 2) around the root s_1, where s_1 is on Γ_4 and $f(s_1) = 0$. The circle can be chosen such that (i) the only root of $f(s)$ inside the circle is s_1, and (ii) it is small enough so that the integral along the lower half-circle is π; that is

$$\int_{\Gamma_{b\to c'\to d}}\arg f(s)ds = \pi. \tag{10}$$

From Theorem 3,

$$\frac{1}{2\pi}\left[\int_{\Gamma_1+\Gamma_2+\Gamma_3}\arg f(s)ds + \int_{a\to b\to c'\to d\to e}\arg f(s)ds\right] = L_1 + 1.$$

Noting (10) and $\int_{a\to b} = \int_{d\to e} = 0$, we obtain (9). □

Given the integration value of (9), it is not always possible to determine L_1 and L_2 uniquely. However, as long as stability is concerned, the exact number of roots is not important; if the integration value of (9) is nonzero, the system is unstable. Hence Theorem 4 can be used instead of Theorem 3 for the stability check.

In order to check the stability of (1) using Theorem 4 and Theorem 1 (or Theorem 2), two problems should be considered.

1. How to compute the integrations in (9)?
2. What if there are roots on L_1, L_2 and L_3?

Firstly, (9) is computed by discretizing the integral path. For example, L_1 is discretized into N segments by $N+1$ points; then the following is satisfied:

$$\int_{\Gamma_1}\arg f(s)ds = \sum_{i=1}^{N}\arg(f(p_i)) - \arg(f(p_{i-1})), \tag{11}$$

as long as

$$\left|\arg(f(p_i)) - \arg(f(p_{i-1}))\right| < 2\pi, \forall i. \tag{12}$$

It is noted that the relation of (11) is not an approximation but an exact equation as long as (12) is satisfied. Hence it is important to discretize the path such that (12) is satisfied. For a circular contour, a guideline for the fixed discretization size is given in [14] based on geometric interpretation and numerical experiences. In the following, a computational procedure is proposed to compute (11). The condition (12) is most likely to be violated when roots are located near the contour; then $\arg f(s)$ changes rapidly. Thus discretization size should be small when roots are near the contour. In the proposed procedure, the discretization size is adaptively adjusted when $|f(s)|$ is

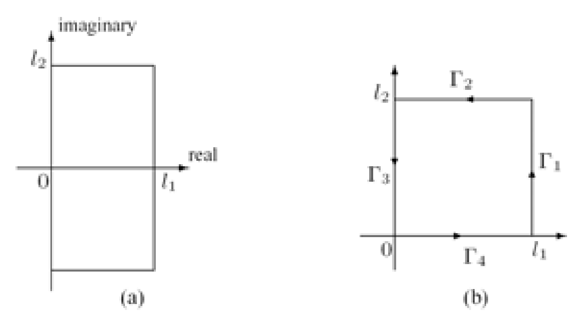

Fig. 1. (a) Closed-right half plane roots bounds, (b) Integration contour.

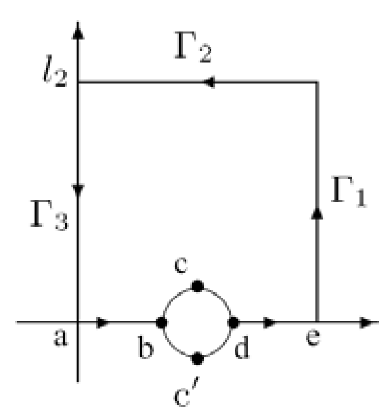

Fig. 2. Integration contour with a small circle around a root on Γ_4.

small (i.e. roots are near the contour). Furthermore, if $|f(s)|$ is smaller than the predetermined value, roots are determined to exist on the contour, which resolves the problem 2

Computational procedure for (9) based on adaptive discretization

1. Let p_i be discretization points of Γ, where p_0 is the starting point of Γ and p_N is the end point of Γ. Then (9) is computed using the following equation:

$$\frac{1}{2\pi}\int_{\Gamma_1+\Gamma_2+\Gamma_3} \arg f(s)ds = \frac{1}{2\pi}\sum_{i=1}^{N} \arg(f(p_i)) - \arg(f(p_{i-1})). \quad (13)$$

It is noted that \$N\$ is determined not in advance but by the next step.

2. Given p_{i-1}, the next point p_i is determined such that

$$|p_i - p_{i-1}| = \min\left(\text{EPS1}, \text{EPS2}|f(p_{i-1})|\right). \quad (14)$$

3. If $f(p_i) < \text{EPS3}$, then stop. There is a root on the contour; hence the system is unstable.

4. Let F_1 be the summation of (13) and F_2 be an integer nearest to $2F_1$. If $|2F_1 - F_2| > \text{EPS4}$, repeat the computation with smaller EPS2 and EPS3.

The third step checks whether roots exist on the contour. The second step is for the condition (12). Although numerical experiences shows that the second step is very reliable, it is not foolproof. Theoretically, it is possible that (12) is violated even if (14) is satisfied. The fourth step is a fail-safe condition to check whether (9) is violated. Note $2F_1$ should be an integer value from (9) when (12) is not violated.

The worst case number of discretization step is determined by EPS1 and EPS2. For example, the worst case step number for Γ_1 is $\Gamma_1/\text{EPS1}$.

EPS values are trade-off parameters between reliability and efficiency of the proposed algorithm. If EPS values are chosen conservatively, the algorithm is reliable (i.e. (12) is less likely violated), but with the sacrifice of efficiency (i.e. it needs more computation). From extensive numerical experiences, EPS values are chosen as in Table 1. In the case of EPS1 and EPS2, smaller values are used for the path Γ_3 since roots can be located near the path Γ_3, while near roots can be avoided for the paths Γ_1 and Γ_2.

Table 1. EPS values in the computational procedure.

	Γ_1 and Γ_2	Γ_3
EPS1	0.1	0.01
EPS2	0.1	0.01
EPS3	10^{-7}	10^{-7}
EPS4	0.2	0.2

Table 2. l_1 and l_2 comparison.

	$A_1 = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}$
l_1(Theorem1)	0.718
l_1(Theorem2)	0.449
l_2(Theorem1)	1.618
l_2(Theorem2)	1.618

	$A_1 = \begin{bmatrix} -2 & -5 \\ 0 & 1 \end{bmatrix}$
l_1(Theorem1)	4.565
l_1(Theorem2)	2.354
l_2(Theorem1)	5.465
l_2(Theorem2)	2.818

4. NUMERICAL EXAMPLES

Example 1: This example compares Theorem 1 and Theorem 2. l_1 and l_2 values given by each theorem for 3 different systems are compared. The 3 different systems are given by

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} x(t) + A_1 x(t-h), \quad (15)$$

where $h = 1$ and A_1 is different for each system. The result is given in Table2. It can be seen that Theorem 2 gives smaller l_1 and l_2 values; thus the bounds of Theorem 2 are less conservative bounds of closed-right half plane roots.

Example 2: Consider (15) with $A_1 = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}$. The system is considered in [15] and is stable for $h \le 6.17$. The computation of (13) is given in Table 3. As seen in Table 3, the computation correctly determines the stability of the system. It is worthwhile noting the change of N of Γ_3 with respect to the change of h, which is the number of the discretization step for the path Γ_3. As h approaches 6.17, roots approach the imaginary axis. To cope with these near roots, the discretization size is adaptively decreased (i.e. N is increased) as proposed in the computational procedure in Section III.

Example 3: This example has a distributed delay

Table 3. Numerical computation of argument principles for different h.

H	1.00	6.16
N of Γ_1	18	18
N of Γ_2	6	6
N of Γ_3	45	265
$\int_{\Gamma_1+\Gamma_2+\Gamma_3} \arg f(s)ds$	0.000	-1.414×10^{-16}

H	6.17	6.18
N of Γ_1	18	18
N of Γ_2	6	6
N of Γ_3	306	278
$\int_{\Gamma_1+\Gamma_2+\Gamma_3} \arg f(s)ds$	-1.414×10^{-16}	1.000

term. Consider the following system:

$$\dot{x}(t) = A_0x(t) + A_1x(t-5) + \int_{-5}^{0} Re^{Pr}Qx(t+r)dr,$$

where A_0, A_1, P, Q, and R are given by

$$A_0 = \begin{bmatrix} 0.000 & 1.000 \\ -0.420 & -1.193 \end{bmatrix},\ A_1 = \begin{bmatrix} 0.000 & 1.000 \\ -0.420 & -1.193 \end{bmatrix},$$
$$P = \begin{bmatrix} 0.702 & 0.323 \\ -0.323 & 0.702 \end{bmatrix},$$
$$Q = \begin{bmatrix} -0.029 & 0.001 \\ 0.194 & 0.312 \end{bmatrix},\ R = \begin{bmatrix} 0.000 & 0.000 \\ -0.597 & -0.552 \end{bmatrix}.$$

This system is the closed-loop system example stabilized by the LQ regulator in [1]. Using Lemma 1, it is found that

$$\left\| \int_{-h}^{0} Re^{Pr}Qx(t+r)dr \right\| \le 0.413.$$

Using Theorem 2, it is found that $l_1 = 0.558$ and $l_2 = 1.226$. The computation of (13) gives 7.068×10^{-17}; hence the system is stable.

5. CONCLUSION

This paper has proposed a numerical method to check the stability of a general class of time delay systems. The proposed method is based on argument principles and the fact that roots whose real values are nonnegative can only exist in the confined area. Using the proposed method, the stability of a general class of time delay systems can be checked reliably and efficiently.

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