A Unified Approach to Exact, Approximate, Optimized and Decentralized Output Feedback Pole Assignment

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Abstract: The paper proposes a new formulation of the output feedback pole assignment problem. In this formulation, a unified approach is presented for solving the pole assignment problem with various additional objectives. These objectives include optimizing a variety of performance indices, and imposing constraints on the output feedback matrix structure, e.g. decentralized structure. Conditions for the existence of the output feedback are discussed. However, the thrust of the paper is on the development of a convergent pole assignment algorithm. It is shown that when exact pole assignment is not possible, the method can be used to place the poles close to the desired locations. Examples are provided to illustrate the method.

Keywords: Linear systems, pole assignment, optimization.

1. INTRODUCTION

A well established technique for the design of linear multivariable time-invariant systems is pole assignment. This is due to the fact that the stability and dynamic behavior of such systems are governed mainly by the pole locations of the closed-loop system. The first important results in pole assignment are reported in [1,2] proving that for an m-input, r-output system of order n, $\min(n, m+r-1)$ closed-loop poles can be assigned by static output feedback provided some mild conditions are satisfied. Methods to find the required feedback matrix are given in [3-5]. Many papers have dealt with sufficient or necessary conditions for the existence of the feedback matrix with varying degree of generality e.g., [6-9]. Procedures are given in [9,10] to assign $\min(n, mr)$ poles by static output feedback, which allows complete pole assignment for higher order multi-input multi-output systems since $mr > (m+r-1)$ for these systems. In [10], the developments are in the state space framework and are based on an incremental method while [11] uses transfer function system description and is based on the exterior algebra.

The solution to the output pole assignment problem, when it exists, is in general non-unique. This is especially true when $mr > n$. In these cases, the freedom in the choice of the output feedback matrix

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can be exploited for the optimization of a design objective [12-15]. The optimization is usually taken as the minimization of the sensitivity of the closed-loop poles to perturbation or uncertainty in the system parameters. This is usually referred to as robust pole assignment (see [16] for a survey). More recently, certain new robustness measures are introduced in [17].

Approximate pole assignment is considered when exact pole assignment is not possible due to, for example, the excess of system order relative to the number inputs and output, e.g., when $n > mr$ (see e.g., [18]). A procedure that involves solving certain matrix equations is given in [19] to assign poles at approximate locations by first assigning the $(n-r)$ eigenvectors. Reference [20] utilizes a gradient flow approach and neural computing for robust approximate pole assignment. This problem is considered in [21] using a genetic algorithm approach where an objective function is optimized.

When the structure of the feedback matrix is constrained, e.g. due to decentralized information flow, the pole assignment problem becomes more complex. This problem has been studied by several researchers going back to the early work on stabilizability [22] to the solvability of pole assignment [23] and procedures for computing the decentralized feedback matrix [10], and finally to the most recent paper on low order compensators [24], just to mention a few.

In this paper, we propose a new formulation and solution to the static output feedback pole assignment problem. In this formulation, a differential approach is adopted in order to find the required feedback via a matrix pseudo-inverse solution. The freedom that exits in the pseudo-inverse solution is exploited to achieve optimization goals. The proposed approach has

several features, including a simple formulation, ability to perform pole assignment with structurally constrained output feedback matrix (e.g., decentralized structure), and the possibility of achieving approximate pole assignment when the conditions for the exact pole assignment are not satisfied. More importantly the main contribution of the paper is to propose a unified approach to all these problems, with a straightforward implementation.

2. PROBLEM FORMUALTION

Consider the controllable and observable linear time-invariant multivariable system

$$\dot{x} = Ax + Bu,$$
$$y = Cx, \tag{1}$$

where x is the $n \times 1$ state vector, u is the $m \times 1$ input vector and y is the $r \times 1$ output vector with $n > \max(m,r)$. The characteristic polynomial of the open-loop system is

$$p(s) = \det(sI - A) = \sum_{i=0}^{n} p_i s^{n-i}, \tag{2}$$

where $p_0, p_1, \cdots, p_n$ are constant coefficients with $p_0 = 1$. Applying the output feedback control law

$$u = -Ky + v, \tag{3}$$

where K is the $m \times r$ output feedback matrix and v is the $m \times 1$ command vector, we obtain the closed loop system

$$\dot{x} = \hat{A}x + Bv,$$
$$y = Cx, \tag{4}$$

where $\hat{A} = A - BKC$ is the closed-loop system matrix. The characteristic equation of the closed-loop system can be written as

$$q(s) = \det(sI - A + BKC) = \sum_{i=0}^{n} q_i s^{n-i}, \tag{5}$$

where $q_0, q_1, \cdots, q_n$ are constant coefficients of the closed-loop characteristic polynomial with $q_0 = 1$. These coefficients are related to the corresponding open-loop coefficients by [10]

$$q_i = p_i + Tr\,[K\,(p_{i-1}CB + p_{i-2}CAB + \cdots + p_0 CA^{i-1}B)] + \psi_i(K), \qquad i = 1, 2, \cdots, n, \tag{6}$$

where Tr denote the trace of the matrix. Note that the trace term is a linear function of the elements of K

whereas $\psi_i(K)$ is a nonlinear function of these elements and consists of minors of orders 2 to $\max(m,r)$ of K [10]. Expressing the trace as product of two vectors, (6) becomes

$$q_i = p_i + (p_{i-1}e_0 + p_{i-2}e_1 \cdots + p_0 e_{i-1})k + \psi_i(K), \tag{7}$$
$$i = 1, 2, ..., n,$$

where e_i are $1 \times mr$ vectors formed by placing the rows of $CA^{i-1}B$ next to one another, and k is an $mr \times 1$ vector formed by stacking the columns of K.

The pole assignment problem is to find the $m \times r$ matrix K, or equivalently the $mr \times 1$ vector k such that the coefficients of the characteristic polynomial change from the initial (open-loop) values $p = (p_1\ p_2 \cdots p_n)^T$ to the desired final values denoted by $d = (d_1\ d_2 \cdots d_n)^T$. We solve this problem by incrementally moving from p to d over a smooth time trajectory such as a cycloid defined by

$$q_{des}(t) = \begin{cases} p + \left(\dfrac{t}{T} - \dfrac{1}{2\pi}\sin\dfrac{2\pi}{T}t\right)(d - p) & t \le T \\ d & t > T, \end{cases} \tag{8}$$

where $q_{des}(0) = p$, $q_{des}(T) = d$ and T is the duration to move from initial to the desired values. The choice of the particular trajectory (8) is due to its important smoothness property that its first, second and higher order derivatives are all continuous and finite. This results in smooth transition from the initial to desired values of the characteristic coefficient vectors. Higher values of T correspond to slow variations in q_{des} which generally improve stability of the error equation to be defined in Section 4. However, too large values of T slow down response. Suppose that we are at an intermediate point at time t_{int} along the trajectory where a feedback matrix K has already been applied and the intermediate characteristic coefficient vector is $q(t_{\text{int}}) \equiv (q_1\ q_2 \cdots q_{n-1})^T$. We want to move incrementally to the next point along the trajectory by applying incremental feedback matrix δK. Using (7), the change in the coefficients of the characteristic polynomial is

$$q_i + \delta q_i = q_i + (q_{i-1}f_0 + q_{i-2}f_1 \cdots + q_0 f_{i-1})\delta k + \psi_i(\delta K), \qquad i = 1, 2, \cdots, n, \tag{9}$$

where f_i are $1 \times mr$ vectors formed from the rows of $C\hat{A}^{i-1}B = C(A + BKC)^i B$, and $\psi_i(\delta K)$ are nonlinear functions containing second and higher order product terms in the elements of the incremental

matrix δK and thus $\psi_i(\delta K) \approx 0$. Equation (9) can now be written as

$$
\begin{pmatrix} \delta q_1 \\ \delta q_2 \\ \vdots \\ \delta q_n \end{pmatrix} = \begin{pmatrix} q_0 & 0 & \cdots & 0 \\ q_1 & q_0 & \cdots & 0 \\ \vdots & & & \vdots \\ q_{n-1} & q_{n-2} & \cdots & q_0 \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_{n-1} \end{pmatrix} \delta k \tag{10}
$$

or

$$
\delta q = Q(k) F(k) \delta k \equiv J(k) \delta k, \tag{11}
$$

where $q_0 = 1$, $Q(k)$ and $F(k)$ are $n \times n$ and $n \times mr$ matrices defined in (10), and $J(k) \equiv Q(k)F(k)$ is the $n \times mr$ Jacobian matrix, and due to its particular structure $rank(Q(k)) = n$ for all k (or all K). The incremental (11) can be expressed as a differential equation by dividing both sides by δt to get

$$
\dot{q} = J(k)\dot{k}. \tag{12}
$$

The above is a differential equation relating the characteristic polynomial vector to the feedback vector. Equation (12) can be solved for $\dot{k}$ using the generalized pseudo-inverse method

$$
\dot{k} = J^*(k)\dot{q}_{des} + \alpha(I_{mr} - J^*(k)J(k))h, \tag{13}
$$

where $J^*(k)$ is the $mr \times n$ pseudo-inverse matrix of $J(k)$, q_{des} is the desired closed-loop characteristic polynomial vector, α is an arbitrary scalar, h is an arbitrary $mr \times 1$ vector and I_{mr} is the $mr \times mr$ identity matrix. The rank of $J(k)$ and the particular choice of the free vector h play important roles in pole assignment and optimization, which will be discussed shortly. Note that due to the particular choice of q_{des} given by (8), its derivative needed in (13) is continuous. Integration of (13) will give $k(t)$ whose steady-state value will be the required feedback vector. However, we must first discuss the condition under which the steady-state exits.

3. PSEUDO-INVERSE SOLUTIONS

The rank of the Jacobian matrix $J(k)$ determines the existence and the accuracy of the solution (13). Now $rank(J(k)) = rank(Q(k)F(k)) = rank(F(k))$, where $F(k)$ is an $n \times mr$ matrix formed from the rows of $C(A + BKC)^i B$, i.e., $F = (f_0\ f_1 \cdots f_{n-1})^T$, as explained before. Since the characteristic coefficient trajectory vector changes with time from the initial value to the desired value, the value of the feedback matrix K (or the feedback vector k) also changes

during this transition. As a result, the rank of the Jacobian matrix is dependent on K and ultimately on the chosen trajectory $q_{des}(t)$.

Suppose that $n \times mr$ matrix $F(k)$ and consequently $J(k)$ have full rank n for all full rank K except possibly for a finite number of K matrices. In this case we say that the *generic rank* of $F(k)$. or J(k) is *n*, and write it as $grank(J(k)) = grank(F(k)) = n$. The generic rank can be found by applying a randomly generated feedback matrix to the system, and evaluating the rank of the resulting matrix $F(k)$. It is noted that the K matrices that make $F(k)$ or $J(k)$ rank deficient are the solution to $\det(F(k)F^T(k)) = 0$ which is polynomial equation in the mr elements of K. If $grank(F(k)) = n$, the polynomial vanishes only at finite isolated points. If the chosen trajectory $q_{des}(t)$ causes the system to pass through one or more of these isolated points, $F(k)$ becomes rank deficient. In this case, we can apply an arbitrary (random) initial feedback matrix K_0 to the system to obtain a new system thus effectively changing the initial characteristic coefficient vector *p*. The trajectory is now different except for the final value, which is the desired characteristic polynomial vector *d*. Thus unless this final value causes rank deficiency, it will be possible to perform exact pole assignment. Note that it is advisable to scale the initial output matrix K_0 such that $\|BK_0C\|$ is of the same order as $\|A\|$ so that K_0 does not either overwhelm the system characteristics or causes insignificant change. In the exceptional case that the rank deficiency of $J(k)$ is due to the particular desired characteristic polynomial vector *d*, one can place the poles close to, but not exactly at, the desired locations.

It is also noted that in practice evaluating $J(k)$ using the matrices $C(A + BKC)^i B$, $i = 0, 1, 2, \cdots, n-1$ can result in ill-conditioned cases and exponential computation complexity since an $n \times n$ matrix is raised to the power of n. Therefore, $J(k)$ is computed using (11), i.e., elements of the feedback matrix are changed incrementally, one at time, and the resulting changes to closed-loop characteristic polynomial coefficients q_i are found using an efficient method. The Jacobian matrix is then computed by forming the ratio of the changes in q_i to the changes in each of the feedback matrix elements.

Although it is theoretically possible to integrate (13) to determine the feedback matrix, in practice numerical difficulties such as integration drift due to small inaccuracies or ill-conditioned matrices can cause instability of the solution. In order to circumvent these problems and to make the solution robust, we define the error characteristic coefficient $\varepsilon(t)$ as

$$\varepsilon(t) = q_{des}(t) - q(t), \qquad (14)$$

where the desired characteristic coefficient trajectory $q_{des}(t)$ is given by (8) and the actual characteristic vector $q(t)$ is obtained from the coefficients of $\det(sI - A + BKC)$, with K being the last value computed. We now apply a "computation feedback" so that (13) is modified to

$$\dot{k} = J^*(\dot{q}_{des} + G\varepsilon) + \alpha(I_{mr} - J^*J)h, \qquad (15)$$

where G is the $n \times n$ positive definite computation feedback matrix which for simplicity can be chosen as a diagonal matrix with positive elements. Note that due to the particular choice of the trajectory in (8), $\dot{q}_{des}$ is finite. We refer to (15) as the companion dynamic system. Note that for simplicity of notation the dependency of J on k is dropped in (15), i.e., $J \equiv J(k)$.

Pre-multiplying both sides of (15) by J and using (12), we get

$$\dot{q} = JJ^*(\dot{q}_{des} + G\varepsilon) + \alpha J(I_{mr} - J^*J)h. \qquad (16)$$

We form $\dot{\varepsilon} = \dot{q}_{des} - \dot{q}$, and substitute ε from (14) into the resulting equation to get

$$\dot{\varepsilon} = -JJ^*G\varepsilon + (I_n - J^*J)\dot{q}_{des} \\ - (\alpha J(I_{mr} - J^*J)h. \qquad (17)$$

Since we assumed that $rank(J) = n$, we have $JJ^* = I_n$ and $J(I_{mr} - J^*J) = J - JJ^*J = 0$. Furthermore, we if we choose $G = gI_n$ where g is a constant positive scalar, (17) simplifies to

$$\dot{\varepsilon} = -g\varepsilon \qquad (18)$$

or

$$\varepsilon(t) = e^{-gt}\varepsilon_0, \qquad (19)$$

where $\varepsilon_0 = \varepsilon(0)$ is the initial error vector. It is seen that the errors decrease monotonically (i.e., the error dynamics are asymptotically stable), and thus the desired characteristic coefficient vector is achieved in the steady-state. The required output feedback is the steady state value of $k(t)$ that is obtained by solving the companion dynamic system (15) using for example Matlab/simulink.

Considering the above development and the earlier discussion regarding the rank of the Jacobian matrix for the existence of a solution, we can state the following theorem.

Theorem 1: Consider the *m*-input, *r*-input system of order *n* given in (1) and the static output feedback

law (3). Then it is possible to assign min(n, mr) poles at or close to the desired locations provided that $grank(J) = n$.

Note that the condition $grank(J) = n$ is equivalent to that of

$$grank(F) = grank(f_0^T, f_1^T, \cdots f_{n-1}^T) = n, \qquad (20)$$

where f_i are $1 \times mr$ vectors formed from the rows of $C(A + BKC)^i B$, and K is a randomly generated $m \times r$ matrix. The condition $grank(J) = n$ is more convenient to determine numerically whereas $grank(F) = n$ provides better insight into the role of systems matrices in pole assignment.

It is noted that when $mr = n$, the second term in (13) becomes zero and h has no effect on the feedback matrix. Provided that the rank condition of the above theorem is satisfied, the pole assignment problem has at least one solution, and the required K can be obtained using the above procedure. However, in general the solution is not unique. The solutions can usually be found by applying different (randomly generated) initial feedback matrices to the open-loop system to obtain a new system. The above procedure can then be applied to these new systems to obtain the desired characteristic polynomial. However, there is no guarantee that all real solution can be found unless a large number of different initial feedback matrices are tried.

4. ADDITIONAL OBJECTIVE AND CONSTRAINTS

In this section, we discuss how additional objectives such as optimization of a performance index, or imposing restriction on the output feedback structure can be achieved within the framework of the above formulation.

4.1. Optimization

It is noted from (13) that in case $mr > n$ and $grank(J) = n$, the feedback vector *k(t)* will be dependent on the choice of h. In this case, the freedom resulting from extra number of parameters $(mr - n)$ can be used for optimization. Suppose that we want to minimize the performance index described by a function $\phi(k)$. We now show that

$$h = -\frac{\partial \phi(k)}{\partial k} \qquad (21)$$

will achieve the minimization. To this end, we find the time derivative of $\phi(k)$

$$\dot{\phi}(k) = \left(\frac{\partial \phi(k)}{\partial k}\right)^T \dot{k}$$

$$= -h^T \left(J^*(\dot{q}_{des} + G\varepsilon) + \alpha(I_{mr} - J^*J)h\right) \quad (22)$$

$$\leq -\alpha h^T (I_{mr} - J^*J)h$$

$$+ \left\|h^T J^*\right\| \left(\|(\dot{q}_{des})_{max}\| + g\|\varepsilon\|_{max}\right),$$

where (15) has been substituted for $\dot{k}$. The maximum value of $\dot{q}_{des}$ is found from (8) to be $(\dot{q}_{des})_{max} = \frac{1}{T}(d-p)$ where $(d-p)$ is the difference between the desired and initial (open-loop) characteristic polynomial coefficient vectors. Furthermore, $(I_{mr} - J^*J)$ is both symmetric and idempotent, $G = g I_n$, and $\|\varepsilon\|_{max} = \|\varepsilon_0\|$ as seen by (19). As a result

$$\dot{\phi}(k) \leq -\alpha \, h^T (I_{mr} - J^*J)(I_{mr} - J^*J)^T h$$

$$+ \left\|h^T J^*\right\| \left(\frac{\|d-p\|}{T} + g\|\varepsilon_0\|\right). \quad (23)$$

The first term on the right hand side of (23) is negative definite and its magnitude can be made as large as desired by choosing a large value for α. On the other hand the second term can be made as small as desired by increasing the value of the trajectory time T. Finally, the third term containing $\|\varepsilon_0\|$ can be made small by choosing the initial characteristic coefficient vector to be close to its open-loop value. Consequently, the left hand side of (23) is guaranteed to be negative by choosing sufficiently large values of α and T, and thus the minimization of $\phi(k)$ is achieved. For maximization of the performance index, the free vector is chosen as $h = \frac{\partial \phi(k)}{\partial k}$.

We now give examples of possible optimizations goals while achieving pole assignment.

(a) Consider pole assignment with the minimum norm of the feedback matrix. In this case, we desire to find among infinite number of solution for K the one with minimum norm so that high gain is avoided and un-modelled dynamics are not excited. In this case, we set $\phi(k) = \frac{1}{2}\|k\|^2$, and $h = -\frac{\partial \phi(k)}{\partial k} = -k$.

(b) Suppose we want to find K so that pole positions are least sensitive to changes or uncertainty in the system parameters, e.g., A, B, C. It has been shown in [26] that the minimization of the function

$$\phi = \|A - BKC\|$$

achieves the closed-loop pole minimum sensitivity objective. It can easily be verified that

$$\phi = \|A - BKC\| = Tr[(A - BKC)^T (A - BKC)].$$

Expanding the expression inside the trace, taking the derivative $\partial \phi(k)/\partial K$ and noting that

$$\frac{\partial}{\partial K} Tr(UKV) = U^T V^T$$

for any two matrices U and V with appropriate dimensions, we obtain

$$\frac{\partial \phi}{\partial K} = B^T (A - BKC) C^T.$$

The $mr \times 1$ free vector h to be used in (13) is now formed by stacking the columns of the matrix $-B^T (A - BKC) C^T$ where the negative sign is due to minimization (see (21)).

(c) A controllability measure for a system is introduced in [25] which quantify the ease of controlling a system. An optimization objective can be maximizing the controllability measure of the closed-loop system, in addition to pole assignment. It is shown that the controllability measure is [25]

$$\phi = \min_i \left\|C u_i(K) v_i^+(K) B\right\| \quad i = 1, 2, \cdots, n, \quad (24)$$

where $u_i(K)$ and $v_i(K)$ are the right and left eigenvectors associated with the closed-loop pole at λ_i, and the superscript denotes the conjugate transpose. Thus to achieve maximum controllability measure, we evaluate $\frac{\partial \phi}{\partial K}$ numerically using (24), and set $h = \frac{\partial \phi}{\partial K}$.

The above three cases are just a few examples of possible performance index $\phi(k)$, and other performance indices such as those in [17,26] can be similarly formulated. It must be noted that the procedure discussed above yields a feedback matrix that is optimal for the given performance index and the chosen trajectory $q_{des}(t)$.

Example 1: Consider the 5-th order 3-input, 2-output system

$$\dot{x} = \begin{pmatrix} -0.4 & 0.2 & 0.6 & 0.1 & -0.2 \\ 0 & -0.5 & 0 & 0 & 0.4 \\ 0 & 0 & -2 & 0 & 0.2 \\ 0.2 & 0.1 & 0.5 & -1.25 & 0 \\ 0.25 & 0 & -0.2 & 0.5 & -1 \end{pmatrix} x$$
$$+ \begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{pmatrix} u,$$
$$y = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \end{pmatrix} x.$$

It is desired to place the closed loop poles at $-1, -1, -2, -3, -4$ which corresponds to the desired characteristic polynomial vector $d = (11 \quad 45 \quad 85 \quad 74 \quad 24)$. It is also desired to minimize the magnitude of the feedback matrix, i.e., $\min\{\|K\|\}$. Here, there is one degree of freedom since $n = 5$ and $mr = 6$. Using the above procedure with $g = 10$, $\lambda = 1$ and $T = 10$s, we find

$$K = \begin{pmatrix} 0.9326 & -1.11 \\ 0.04534 & -0.9712 \\ 3.632 & -4.106 \end{pmatrix}.$$

The magnitude of K is $\|K\| = 5.753$. By evaluating $\det(sI - A + BKC)$, it can be verified that the desired closed-loop poles are obtained. In fact the magnitude of the error of the characteristic coefficient vector is $\|\varepsilon\| = 4.1 \times 10^{-7}$ after $t = 12 s$ and will eventually go to zero. Note that if we turn off the optimization by setting $\alpha = 0$, we can obtain many different K matrices by applying different initial feedback matrices K_0, all of which place the closed-loop poles at the desired locations. Two such matrices are given below

$$K_1 = \begin{pmatrix} 1.465 & -0.2812 \\ 2.305 & -0.5875 \\ 9.896 & 0.1114 \end{pmatrix}, \|K_1\| = 10.29;$$

$$K_2 = \begin{pmatrix} -3.42 & 4.309 \\ 8.555 & -10.21 \\ -4.973 & 5.344 \end{pmatrix}, \|K_2\| = 16.15.$$

These other solutions have higher norms than the one obtained by optimization.

4.2. Approximate pole assignment

It is instructive to investigate the case where $grank(J) = mr < n$. This case occurs when the number of parameters in the feedback matrix is less than the number of poles. Since now $JJ^* \neq I_n$ but $(I_{mr} - J^*J) = 0$ as before, (17) becomes

$$\dot{\varepsilon} = -JJ^*G\varepsilon + (I_n - JJ^*)\dot{q}_{des}. \tag{25}$$

In order to determine the stability and solution to (21), we consider the Lyapunov function candidate

$$V = \frac{1}{2}\varepsilon^T\varepsilon, \tag{26}$$

whose derivative along the trajectory of (25) is

$$\dot{V} = \varepsilon^T\dot{\varepsilon} = -g\,\varepsilon^T\,J\,J^*\varepsilon + \varepsilon^T(I_n - JJ^*)\dot{q}_{des}, \tag{27}$$

where $G = gI_n$ has been substituted in (27). Since JJ^* is a symmetric positive semi-definite matrix, it can be expressed as $JJ^* = H^TH$ where H is a positive semi-definite matrix with $rank(H) = rank(J)$. Furthermore, it can be verified that $\|I_n - JJ^*\| = b$ where b is a binary number such that $b = 0$ if $rank(J) = n$, and $b = 1$ if $rank(J) < n$. Finally, it is evident from (8) that $\|(\dot{q}_{des})_{max}\| = \frac{1}{T}\|d - p\|$. With these in mind, (27) is expressed as

$$\dot{V} \leq -g\|H\varepsilon\|^2 + \frac{b}{T}\|d - p\|\|\varepsilon\|. \tag{28}$$

Note that when $grank(J) = n$, we have $JJ^* = I_n = H$, $b = 0$, and $\dot{V} < 0$, i.e., $\dot{V}$ is negative definite, thus the error dynamic is asymptotically stable meaning that the actual characteristic polynomial vector becomes equal to its desired value, as shown before. However, in case $grank(J) = mr < n$, one would expect that if mr is "close" to n and if the gain g and the trajectory time T are chosen sufficiently large, then the first term in (28) would dominate the second term, and $\dot{V}$ becomes negative semi-definite, i.e. $\dot{V} \leq 0$.

Therefore, T is chosen such that $T \gg \dfrac{\|d - p\|}{g\|H\|}$. Under this condition, the system will be stable in the sense of Lypunov, i.e., the errors are bounded, and the bound depends on the difference $(n - mr)$. This implies that using the proposed method, approximate pole assignment is possible in cases where the difference $(n - mr)$ is small, as the following example demonstrates. In general, however, the errors between desired and acquired poles depend on the generic rank of J, and there is no guarantee that the pole locations

are satisfactory. However, the pseudo-inverse formulation guarantees that the errors are minimum and that the poles are as close as possible to the desired locations.

Example 2: Consider the system *(A,B,C)* with the same *A* as in Example 1, and the following *B* and *C*

$$B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 0 & -1 \\ 0 & 2 \\ 0 & 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 1 \end{pmatrix}.$$

It is desired to place the poles at locations specified in Example 1, i.e., {−1,−1,−2,−3,−4}, with the desired characteristic polynomial $q_{des} = (11\ 45\ 85\ 74\ 24)$. Since *n = 5* and *mr = 4*, arbitrary pole assignment is not possible. However, since *n* and *mr* are close, we expect to be able to place the poles close to the desired locations. Using the procedure given in Section 3, with *g* = 10, and *T* = 20, we find the output feedback matrix *K*, close-loop characteristic polynomial vector q, and the set of closed-loop poles Λ_c as

$$K = \begin{pmatrix} -9.333 & 15.24 \\ -0.4976 & 0.7743 \end{pmatrix},$$

$$q = (11.0045, 45.0135, 85.0243, 74.0154, 24.0041),$$

$$\Lambda_c = \{-0.9983 \pm j0.0267, -2.0209,$$
$$-2.9488, -4.0385\}.$$

Note that instead of poles at { −1, −1} we now have a pair of conjugate complex poles very close to the two desired poles. The other poles are also very close to their desired values. The solution is not unique and other feedback matrix can be found to produce approximate poles locations. The norm of the steady-state error is $\|e\| = 0.0022$.

4.3. Structurally constrained feedback matrix

When the information flow from the output to the input is restricted, such as in decentralized systems, the feedback matrix becomes structurally constrained, e.g., some feedback matrix elements will be zero or fixed. For example, an *N*-station decentralized system is described by

$$\dot{x} = Ax + \sum_{j=1}^{N} B_j u_j , \qquad (28)$$

$$y_j = C_j x, \quad j = 1, 2, \cdots, N,$$

where u_j are $m_j \times r$ local input vectors, y_j are $r_j \times 1$ local output vectors and the feedback matrix

has a block diagonal structure $K_{sc} = block\ diag$ $(K_1, K_2, \cdots, K_N)$ where K_j are $m_j \times r_j$ matrices, and the subscript *sc* stands for structurally constrained.

In the case of constrained feedback matrix, the developments of the previous sections still hold but some elements of *k, F* and *J* become zero and must be removed. Suppose that the $\mu \times 1$ feedback vector *k* is formed from K_{sc} by stacking the columns of K_{sc}, after removing the zero or fixed elements of K_{sc}, where $\mu = mr - \nu$ and ν is the number of fixed elements of *K*. Similarly, the $n \times \mu$ Jacobian matrix J_{sc} is found by making incremental changes to the non-fixed elements of K_{sc} and computing the changes in the characteristic polynomial coefficients, as described before. We can now state the following lemma, which follows from Theorem 1.

Lemma 1: Consider the m-input, r-output system of order n given in (1) and the structurally constrained output feedback matrix K_{sc}. Then it is possible to assign $\min(n, \mu)$ poles at or close to the desired locations provided that $grank(J_{sc}) = n$.

Note that since there are now only μ parameters in the feedback matrix, a necessary condition for pole assignment is $\mu \geq n$. It is also noted that a necessary condition for pole assignability is the absence of fixed modes with respect to the constrained output feedback [27]. Furthermore, it is easy to show that $grank(J_{sc}) < n$ when a system has fixed modes with respect to K_{sc}.

Example 3: Consider a system similar to that of Example 1, but with a two-station decentralized structure as follows

$$\dot{x} = \begin{pmatrix} -0.4 & 0.2 & 0.6 & 0.1 & -0.2 \\ 0 & -0.5 & 0 & 0 & 0.4 \\ 0 & 0 & -2 & 0 & 0.2 \\ 0.2 & 0.1 & 0.5 & -1.25 & 0 \\ 0.25 & 0 & -0.2 & 0.5 & -1 \end{pmatrix} x$$

$$+ \begin{pmatrix} 1 & -1 \\ 2 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 0 \\ 1 \\ -2 \\ 1 \end{pmatrix} u_2,$$

$$y_1 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \end{pmatrix} x, \quad y_2 = \begin{pmatrix} 0 & 0 & 1 & -1 & 1 \end{pmatrix} x.$$

It is desired to place the poles at (−1, −1, −2, −3, −4),

as in Example 1. The decentralized controller matrix has the structure $K_{sc} = \begin{pmatrix} k_1 & k_2 & 0 \\ k_4 & k_4 & 0 \\ 0 & 0 & k_5 \end{pmatrix}$. It is found that for this system $grank(J_{sc}) = 5$, and therefore arbitrary pole assignment is possible. Using the procedure described before, we find one possible feedback matrix as

$$K_{sc} = \begin{pmatrix} -0.9785 & -2.0288 & 0 \\ -1.4546 & -2.2433 & 0 \\ 0 & 0 & 0.5675 \end{pmatrix}.$$

Even though the number of free parameters in K_{sc} is equal to the number of poles, it is possible to have multiple solutions for K_{sc} due to the nonlinear nature of the problem. These solutions can be obtained by applying different initial feedback matrices before starting the procedure. We obtained eight different K_{sc} for this example.

5. DISCUSSION AND CONCLUSIONS

Pole assignment requires solving a set of highly nonlinear polynomial equations in the elements of the output feedback matrix K. In this paper, we have transformed these equations into a companion dynamic system whose steady states values are the elements of the desired K. By defining an error vector and solving the error dynamics of the companion system, conditions are derived for asymptotic convergence of the error dynamics, or equivalently the existence of K for pole assignment.

Let us now compare the proposed method to other approaches to pole assignment. The main feature of the proposed method is that it provides a unified framework for achieving poles assignment with additional objectives and constraints, i.e., exact, approximate, optimized and structurally constrained feedback pole assignment. Unlike most other approaches that require complex computation such as solution to matrix equations or heuristic approaches such as neural and genetic algorithms, the proposed method can be implemented directly and efficiently. In fact we have developed a program in Matlab/Simulink that takes the system matrices (A,B,C) and the desired performance function and provides the feedback matrix, or matrices in cases where several solutions exist. The implementations show zero (numerically insignificant) norm of the error vector between the desired and actual characteristics coefficient vectors when pole assignability conditions given in the paper are satisfied. In other cases where only approximate pole assignment is possible, the norm of the error

depends on the rank of a defined Jacobian matrix. The particular formulation and solution proposed in the paper achieves minimum errors in such cases.

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