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Absolute Stability Margins in Missile Guidance Loop

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Abstract: This paper deals with the stability analysis of a missile guidance loop employing an integrated proportional navigation guidance law. The missile guidance loop is formulated as a closed-loop control system consisting of a linear time-invariant feed-forward block and a time-varying feedback gain. Based on the circle criterion, we have defined the concept of absolute stability margins and obtained the gain and phase margins for the system assuming 1st order missile/autopilot dynamics. The correlation between the absolute stability margins and the margins derived from the frozen system analysis is also discussed.

Keywords: Absolute stability margins, frozen system analysis, integrated proportional navigation guidance, missile guidance loop.

1. INTRODUCTION

Proportional navigation guidance (PNG) is widely used for the terminal homing guidance of missiles having acceleration controller type autopilots because of its simplicity, effectiveness, and ease of implementation. For the missile using an attitude angle controller, which controls heading angle, the integrated proportional navigation guidance (IPNG) can be used with various merits of PNG. One of the advantages to implementing an attitude angle controller is the easiness of controlling the direction of missile velocity vector as a missile approaches its target, which is known as the impact angle. Two major reasons for controlling the impact angle are as follows. With the former, for anti-ship or anti-tank missiles, the terminal impact angle is important for enhancement of warhead effect. In the latter, when a missile has the mission of passing through several waypoints of a given order, a smooth flight path can be obtained by a suitable selection of the impact angle at each waypoint. Note also that IPNG is the integral form of PNG, two guidance laws with different expressions are identical if initial state values in IPNG law are the same. In this work, we assume that the missile system is based on an attitude angle controller and IPNG law.

Many works [1-5] have been conducted on the subject of stability analysis of missile guidance loop using PNG. Guelman [1] investigated the finite time absolute stability of PNG systems by employing the Kalman-Yakubovitch-Popov lemma. In Tanaka *et al.* [2,3], the concept of hyperstable range was introduced and the stable time region for missile guidance loop using PNG command was derived. Rew *et al.* [4] applied practical stability methods to derive a lower bound on the time-to-go for a PNG system with single time constant dynamics. Gurfil *et al.* [5], by using the well-known circle criterion [6,7], established an analytic bound for the time of flight up to which stability can be assured. As the works for guidance stability of missiles with model uncertainties, Impram *et al.* [8] suggested the robust circle criterion and the robust Popov criterion that had been extended to systems involving both structured and unstructured uncertainties in the linear plant. Furthermore, Weiss *et al.* [9] studied the stability of modern guidance laws when the missile's actual model differs from the model used in the design, the analysis of which was performed by means of Lyapunov functions and by means of the multivariable circle criterion.

This paper suggests the concept of absolute stability margins in missile guidance loop using IPNG law. Although the concept of the "absolute-stability gain margin" was also introduced by Hagiwara *et al.* [10], we define not only the absolute stability gain margin but also the phase margin of the IPNG loop by using the circle criterion, and show their resemblance to the results obtained from the frozen system analysis [1,11].

This paper is organized as follows. A mathematical model of IPNG loop is described in Section 2. Section 3 suggests the stability regions be obtained by applying different criteria to the system. Section 4 introduces the concept of absolute stability margins and presents some illustrative results. Concluding remarks are given in Section 5.

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2. GUIDANCE LOOP MODEL

Consider a two-dimensional missile-target engagement geometry as shown in Fig. 1, where the missile M with velocity V_M and the target T with velocity V_T are treated as point masses, R is the relative range between missile and target, θ is line of sight (LOS) angle with respect to a reference line, and θ_M and θ_T are flight-path angles of the missile and the target, respectively.

The kinematic relation between missile and target motions is obtained by resolving velocity components of the missile and the target along and normal to the LOS.

$$\dot{R} = V_T \cos(\theta_T - \theta) - V_M \cos(\theta_M - \theta) \triangleq -V_C, \quad (1)$$

$$R\dot{\theta} = V_T \sin(\theta_T - \theta) - V_M \sin(\theta_M - \theta), \quad (2)$$

where V_C represents the closing velocity.

Assuming that $\theta_M \cong \theta$, $V_M \gg V_T$ and $R(t_f) = 0$ where t_f t_f denotes the total flight time of the engagement, we obtain the expression for the line of sight rate to be

$$\dot{\theta} = \frac{1}{t_{go}}(\theta - \theta_M), \quad (3)$$

where $t_{go} \triangleq t_f - t$ is the time-to-go for the missile to intercept the target.

The IPNG law for a missile having an autopilot of attitude angle controller type is defined as

$$\theta_C = N\theta, \quad (4)$$

where N is called the navigation constant.

The missile/autopilot (M/AP) dynamics can be expressed generally in the following transfer function form assuming a linear missile dynamics

$$G(s) = \frac{\theta_M(s)}{\theta_C(s)} = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}, \quad (5)$$

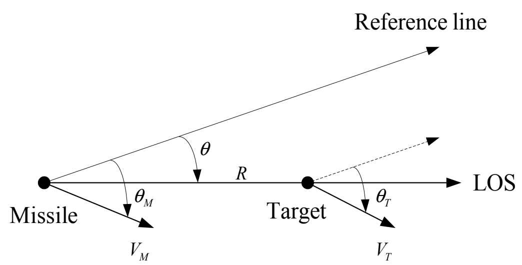

Fig. 1. Missile-target engagement geometry.

where we assume that $\theta_M(s)$ and $\theta_C(s)$ are coprime and $G(0) = 1$, i.e., $b_0 = a_0$. The transfer function $G(s)$ can be realized in the phase-variable canonical form with no loss in generality

$$\dot{x} = Ax + b\theta_C, \quad (6)$$

$$\theta_M = c^T x, \quad (7)$$

where $G(s) = c^T(sI - A)^{-1}b$ and x is the system state and A, b and c^T are defined by

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & \cdot & \cdot & -a_{n-1} \end{bmatrix}, \quad (8)$$

$$b = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix}, \quad (9)$$

$$c^T = \begin{bmatrix} b_0 & \cdot & \cdot & \cdot & b_{n-1} \end{bmatrix}. \quad (10)$$

Let us define a new state vector z as

$$z = \dot{x}. \quad (11)$$

Then x is rewritten in terms of z and θ_C as

$$x = A^{-1}z - A^{-1}b\theta_C. \quad (12)$$

The substitution (12) in (6) yields

$$\dot{z} = Az + b\dot{\theta}_C. \quad (13)$$

Defining $\varepsilon \triangleq \dot{\theta}_C$ and using (3), (4), (6), and (7), we obtain

$$\varepsilon = -\frac{Nc^T A^{-1}z - (Nc^T A^{-1}b + 1)\theta_C}{t_{go}}. \quad (14)$$

Defining $\sigma \triangleq Nc^T A^{-1}z - (Nc^T A^{-1}b + 1)\theta_C$ we rewrite Eqs. (13) and (14) as

$$\dot{z} = Az + b\varepsilon, \quad (15)$$

$$\dot{\theta}_C = \varepsilon, \quad (16)$$

$$\sigma = h^T z + \Gamma\theta_C, \quad (17)$$

$$\varepsilon = -\frac{1}{t_{go}}\sigma, \quad (18)$$

where

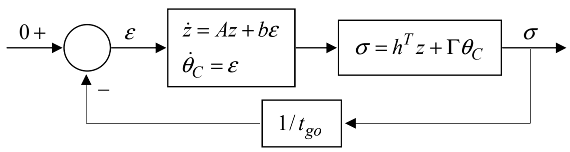

Fig. 2. IPNG loop block diagram.

$$h^T \triangleq N c^T A^{-1}, \tag{19}$$

$$\Gamma \triangleq -(N c^T A^{-1} b + 1). \tag{20}$$

The closed-loop configuration block diagram for the linearized two-dimensional IPNG system is shown in Fig. 2. This linear time-varying system consists of a linear time-invariant element in the forward path and a time-varying gain in the feedback. The linear time invariant portion is

$$H(s) = \frac{\sigma(s)}{\varepsilon(s)} = h^T (sI - A)^{-1} b + \frac{\Gamma}{s} \tag{21}$$

and the feedback is the kinematic gain $1/t_{go}$.

From the inverse matrix form of A and $G(0) = 1$, we can easily show that

$$\Gamma = N - 1 \tag{22}$$

and

$$H(s) = \frac{1}{s}(N G(s) - 1). \tag{23}$$

3. ABSOLUTELY STABLE REGIONS

The IPNG loop system expressed in Fig. 2 falls into categories of Lur'e problem [12] and its stability information could be obtained through a well-known absolute stability theory [12,13].

First, to find the conditions under which the system remains stable, we suppose that the M/AP dynamics is modeled as a simple first-order system

$$G(s) = \frac{1}{T s + 1}, \tag{24}$$

where T is a time-constant of M/AP dynamics.

Applying the well-known Routh-Hurwitz criterion to the linearized IPNG loop under the assumption of frozen time, the conditions for the IPNG loop remain stable and are readily obtained as follows

i) $N > 1$, (25)

ii) $t_{go} > T \quad or \quad SR_H = (T, \infty)$, (26)

where SR_H denotes the stable region by frozen system analysis, the so-called Hurwitz region for system stability, and these are the only necessary conditions for system stability.

According to the method suggested by Guelman [1] which is stated in terms of Popov criterion, sufficient conditions for absolute stability of the IPNG system are derived as

i) $N > 1$, (27)

ii) $t_{go} > NT \quad or \quad SR_P = (NT, \infty)$, (28)

where the stable region denoted by SR_P is said to be Popov region and it is assumed that IPNG starts from the infinite time-to-go. This assumption may not be practical in an actual situation.

For the stability analysis of a missile guidance loop which is only defined over a finite time interval, using the circle criterion, we obtained the sufficient conditions for absolute stability of the IPNG loop with 1st order M/AP dynamics as follows

i) $N > 1$, (29)

ii) $\beta_0 < t_{go} \le \alpha_0 \quad or \quad SR_C = (\beta_0, \alpha_0]$, (30)

where the stable region obtained by circle criterion is expressed as SR_C, $\alpha_0 \triangleq t_f - t_0$ with initial flight time t_0 and β_0 is calculated in terms of α_0, T and N as (see Appendix A)

$$\beta_0 = \frac{T(\Lambda + \Xi)}{(NT - \alpha_0)^2}$$
$$\Lambda \triangleq NT^2 + (N^2 - 4N + 1)T\alpha_0 + N\alpha_0^2 \tag{31}$$
$$\Xi \triangleq -2(N-1)(\alpha_0 - T)\sqrt{NT\alpha_0}.$$

4. ABSOLUTE STABILITY MARGINS

To investigate the stability margin for the IPNG loop with 1st order M/AP dynamics by using absolute stability concepts we first notice that the transfer function and frequency response function of the linear time-invariant portion of the IPNG system are given by

$$H(s) = \frac{(N-1) - T s}{s(T s + 1)}, \tag{32}$$

$$X(\omega) \triangleq R_e H(j\omega) = \frac{-NT}{T^2 \omega^2 + 1}, \tag{33}$$

$$Y(\omega) \triangleq I_m H(j\omega) = \frac{T^2 \omega^2 - (N-1)}{\omega(T^2 \omega^2 + 1)}. \tag{34}$$

For the frozen system in which we assume the time-varying element $\tau \triangleq t_{go}$ as a fixed value, the gain and phase margins at each τ in Hurwitz region can be obtained by employing the definition of stability margin for linear time-invariant systems. The figure for the gain margin at a fixed time point τ in Hurwitz region is shown in Fig. 3. Letting $\tau \triangleq S\,T$; $S \geq 1$, the gain margin is given by

$$GM_F \triangleq 20\log_{10}(\tau / T) = 20\log_{10} S \quad \text{db.} \tag{35}$$

The figure for the phase margin is specified in Fig. 4. From (33) and (34), the frequency ω_0 for $|H(j\omega_0)| = \tau$ is derived with $\tau = S\,T$ as

$$\omega_0 = \frac{\sqrt{-(S^2-1)+\sqrt{S^4+\{4(N-1)^2-2\}S+1}}}{\sqrt{2}\,S\,T}. \tag{36}$$

Then the phase margin is obtained from $X(\omega_0)$ and $Y(\omega_0)$ at ω_0 as follows

$$PM_F = \tan^{-1}\left|\frac{Y(\omega_0)}{X(\omega_0)}\right|. \tag{37}$$

The absolute stability margins can be introduced in

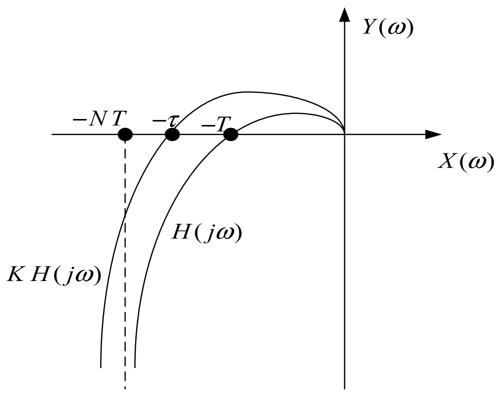

Fig. 3. Gain margin for frozen system.

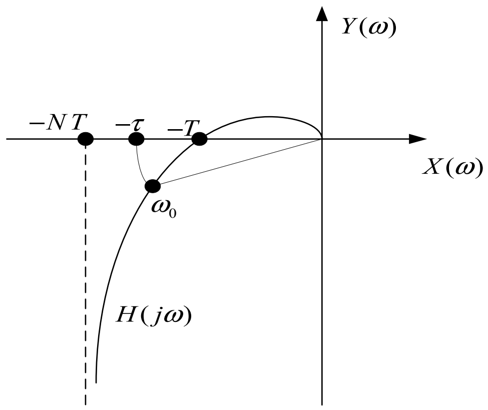

Fig. 4. Phase margin for frozen system.

the same framework as the stability margin concept for linear time-invariant systems, hence we may define the following.

Definition: The absolute stability gain margin and phase margin by the circle criterion are defined as the maximum gain and phase variations of the linear time-invariant element satisfying the circle criterion, respectively.

To seek the absolute stability margins for the IPNG system with 1st order M/AP dynamics, we assume $\tau(t_0) = \alpha_0$ as a fixed value and obtain gain and phase margins at each τ in the stable region $SR_C = (\beta_0, \alpha_0]$ by applying the circle criterion.

Considering Fig. 5, the absolute gain margin at τ in SR_C is obtained as follows. When the Nyquist plot of $KH(j\omega)$ with increasing K meets at one point on the critical circle whose center is located at $-(\alpha_0+\tau)/2$, the maximum gain variation $K_{\max}$ occurs and then the absolute gain margin is given with the 4th order algebraic equation for $K_{\max}$ by (see Appendix B)

$$GM_C \triangleq 20\log_{10} K_{\max} \quad \text{db,} \tag{38}$$

$$\begin{aligned} &T^4 K_{\max}^4 - 2N(\alpha_0+\tau)T^3 K_{\max}^3 \\ &\quad + \{N^2(\alpha_0+\tau)^2 - 2(N^2-4N+1)\alpha_0\tau\}T^2 K_{\max}^2 \\ &\quad - 2N\alpha_0\tau(\alpha_0+\tau)T\,K_{\max} + (\alpha_0\tau)^2 = 0. \end{aligned} \tag{39}$$

In Fig. 6 the graph for the absolute phase margin is indicated in which the absolute phase margin at τ is expressed as γ. The absolute phase margin γ at each τ in SR_C can be obtained with $X(\omega)$ and $Y(\omega)$ in (33) and (34) from the following equations by trial and error.

$$X'(\omega) = X(\omega)\cos\gamma + Y(\omega)\sin\gamma, \tag{40}$$

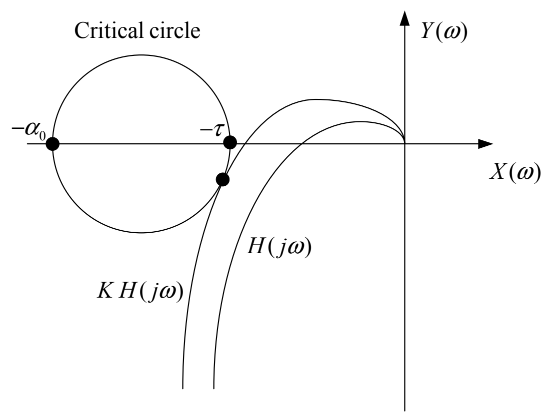

Fig. 5. Gain margin from circle criterion.

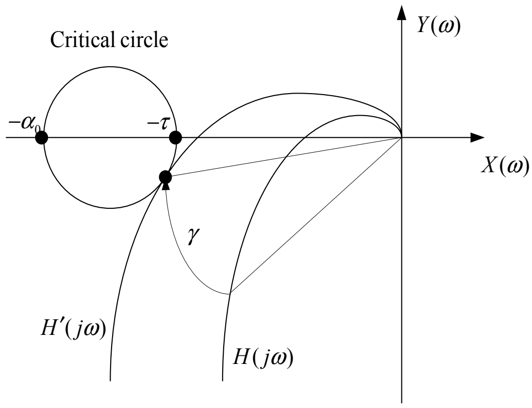

Fig. 6. Phase margin from circle criterion.

$$Y'(\omega) = -X(\omega)\sin\gamma + Y(\omega)\cos\gamma, \quad (41)$$

$$(X'(\omega) + \frac{\alpha_0 + \tau}{2})^2 + Y'(\omega)^2 = (\frac{\alpha_0 - \tau}{2})^2. \quad (42)$$

The absolute gain margin GM_C obtained from (38) and the gain margin GM_F from (35) at each τ in $SR_C = (1.76\,T, 10\,T]$ for $N = 3$ and $\alpha_0 = 10\,T$, are listed in Table 1 and are compared in Fig. 7. The phase margins PM_F from (37) and γ, i.e., PM_C from (40), (41) and (42) are also included in Table 2 and Fig. 8.

From the results, the stability margins from both approaches monotonously decrease with similar shape as the time-to-go τ approaches zero. This observation clearly indicates the general characteristics of PNG-type missile guidance loop as the missile nears intercept.

Table 1. Gain margins in case of $\alpha_0 = 10T$.

τ / T	1.76	2	4	6	8	10
GM_F (db)	4.8	6	12	15.6	18.1	20
GM_C (db)	0	1.4	9.2	14	17.4	20

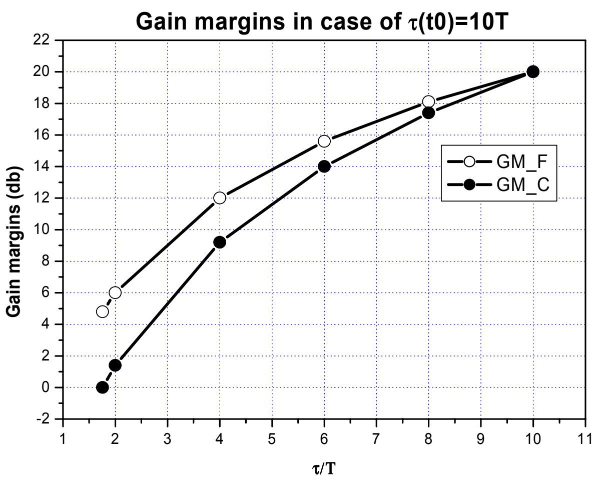

Fig. 7. Graph for gain margins in case of $\alpha_0 = 10T$.

Table 2. Phase margins in case of $\alpha_0 = 10T$.

τ / T	1.76	2	4	6	8	10
PM_F (deg)	22.7	27.6	51.9	63.1	69.3	73.2
PM_C (deg)	0	6	36.5	53.5	64.5	73.2

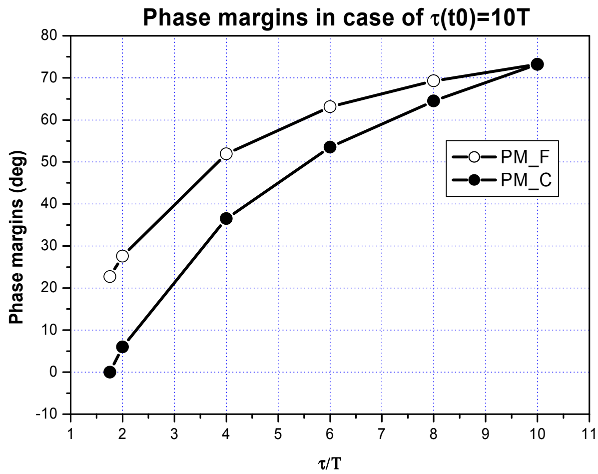

Fig. 8. Graph for phase margins in case of $\alpha_0 = 10T$.

5. CONCLUSIONS

It is well-known that no conclusion can be drawn about stability of nonlinear time-varying systems by employing the frozen system analysis. That is, the conditions provided by frozen system analysis are the necessary, though not sufficient conditions, for system stability. Thus, we may naturally wonder whether the stability margins obtained by the frozen system analysis, which have been widely used for stability analysis of missile guidance loop, are reasonable or not.

By applying the circle criterion, we introduced the concept of absolute stability margins and derived the stability margins for the missile guidance loop assuming 1st order M/AP dynamics and IPNG law. Based on the fact that the circle criterion provides only the sufficient conditions and the frozen system analysis provides the necessary conditions for system stability, it can be concluded that the stability margins of the missile guidance loop considered in this work are located in between the values from the circle criterion and the values by the frozen system analysis.

It may also be noteworthy that the trend and values of the results obtained from the two approaches were very similar to each other, and these results may show to some extent a certain justification of use of the frozen system analysis for stability analysis of missile guidance loop.

APPENDIX A

From (33) and (34), we obtain

$$Y^2 = -\{X(X+T)^2\}/(NT+X). \quad \text{(A1)}$$

The equation of the critical circle which is tangential to $H(j\omega)$ locus and passes through the points $-\alpha_0$ and $-\beta_0$ is given by

$$\{X+(\alpha_0+\beta_0)/2\}^2 + Y^2 = \{(\alpha_0-\beta_0)/2\}^2. \quad \text{(A2)}$$

Canceling Y from (A1) and (A2) gives

$$\{(N-2)T+(\alpha_0+\beta_0)\}X^2 + \{NT(\alpha_0+\beta_0) + \alpha_0\beta_0 - T^2\}X + NT\alpha_0\beta_0 = 0. \quad \text{(A3)}$$

In this case, X has equal roots and its discriminant becomes zero as follows.

$$\{NT(\alpha_0+\beta_0)+\alpha_0\beta_0-T^2\}^2 - 4\{(N-2)T+(\alpha_0+\beta_0)\}NT\alpha_0\beta_0 = 0. \quad \text{(A4)}$$

From (A4), we obtain a quadratic equation in β_0

$$(NT-\alpha_0)^2\beta_0{}^2 - 2T\{NT^2+(N^2-4N+1)T\alpha_0 + N\alpha_0{}^2\}\beta_0 + T^2(N\alpha_0-T)^2 = 0. \quad \text{(A5)}$$

Therefore, (31) can be obtained from (A5) after some algebraic manipulations.

APPENDIX B

From (32), the related equations for $K_{\max}H(j\omega)$ are defined as follows

$$X_1(\omega) \triangleq R_e\, K_{\max}H(j\omega) = \frac{-K_{\max}NT}{T^2\omega^2+1}, \quad \text{(B1)}$$

$$Y_1(\omega) \triangleq I_m\, K_{\max}H(j\omega) = \frac{K_{\max}\{T^2\omega^2-(N-1)\}}{\omega(T^2\omega^2+1)}. \quad \text{(B2)}$$

From (B1) and (B2), we obtain

$$Y_1{}^2 = -\{X_1(X_1+K_{\max}T)^2\}/(NK_{\max}T+X_1). \quad \text{(B3)}$$

The equation of the critical circle which is tangential to $K_{\max}H(j\omega)$ locus and passes through the points $-\alpha_0$ and $-\tau$ is given by

$$\{X_1+(\alpha_0+\tau)/2\}^2 + Y_1{}^2 = \{(\alpha_0-\tau)/2\}^2. \quad \text{(B4)}$$

Canceling Y_1 from (B3) and (B4) gives

$$\{(N-2)K_{\max}T+(\alpha_0+\tau)\}X_1{}^2 + \{NK_{\max}T(\alpha_0+\tau) + \alpha_0\tau - K_{\max}^2T^2\}X_1 + NK_{\max}T\alpha_0\tau = 0. \quad \text{(B5)}$$

In this case, X_1 has equal roots and its discriminant becomes zero as follows.

$$\{NK_{\max}T(\alpha_0+\tau)+\alpha_0\tau-K_{\max}^2T^2\}^2 - 4\{(N-2)K_{\max}T+(\alpha_0+\tau)\}NK_{\max}T\alpha_0\tau = 0. \quad \text{(B6)}$$

The 4th order algebraic equation for $K_{\max}$ given in (39) can be easily obtained by (B6).

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