

## Error Analysis of 3-Dimensional GPS Attitude Determination System

Chansik Park, Deuk Jae Cho, Eun Jong Cha, Dong-Hwan Hwang, and Sang Jeong Lee

**Abstract:** In this paper, the error investigation of a 3-dimensional GPS attitude determination system using the error covariance analysis is given. New efficient formulas for computing the Euler Angle Dilution of Precision (EADOP) are also derived. The formulas are easy to compute and represent the attitude error as a function of the nominal attitude of a vehicle, the baseline configuration and the receiver noise. Using the formula, the accuracy of the Euler angle can be analytically predicted without the use of computer simulations. Applications to some configurations reveal the effectiveness of the proposed method.

**Keywords:** Attitude determination, EADOP, error analysis, GPS.

### 1. INTRODUCTION

In general the GPS receiver provides position, velocity and time information of a vehicle. The attitude can also be determined by measuring the carrier phase from multiple antennas. Due to the bounded characteristics of the attitude error, much attention has been paid to the real-time attitude determination technique [1-4].

To evaluate the accuracy of the obtained positioning and timing result, the expected statistical error magnification factor known as Geometric Dilution of Positioning (GDOP) is generally used. GDOP is defined under the assumption of equal pseudorange measurement error variances in each channel, and represent the geometric contribution of observation errors to the obtained positioning and timing accuracy [5]. GDOP is an indicator of positioning and timing error 'per unit of measurement noise' covariance [6]. GDOP is composed of spatial

and time-related components. The spatial component is called PDOP (Position DOP) and the time-related component is called TDOP (Time DOP). PDOP is further divided into horizontal and vertical factors, HDOP (Horizontal DOP) and VDOP (Vertical DOP).

In the same manner, to predict and analyze the attitude errors, several parameters analogous to the PDOP, HDOP and TDOP concept for the position and time have been defined and introduced. In recent years, Yoon and Lundberg [1] defined a new DOP called the Euler angle dilution of precision (EADOP) for the 3 dimensional attitude determination system. Their derivation is an extension of ADOP (Attitude DOP) in [2] and EADOP includes roll, pitch and yaw elements. The square sum of the elements of EADOP is the same as the square of ADOP. The ADOP and EADOP are efficient tools to attitude error analysis and can be used as a metric of determining a better algorithm [7]. However, ADOP and EADOP do not explicitly show the relations among the attitude error, the nominal attitude of the vehicle and the baseline configuration, so that the computer simulations are required to analyze their affects. The errors in the 2 dimensional attitude determination systems are analyzed in [4] using error covariance analysis and it clearly shows the relations between the attitude error and the nominal attitude, the baseline configuration and the receiver noise.

In this paper, the attitude error covariance matrix of a 3-dimensional attitude determination system is derived. Adding a reasonable assumption to this matrix, new formulas of the EADOP are derived, in which the attitude error is represented as a function of the nominal attitude of a vehicle, the baseline configuration and the receiver noise. Using the formulas, the accuracy of the Euler angle estimate can be predicted analytically while it is predicted through simulations in the previous work [1].

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## 2. ATTITUDE ERRORS IN 3-DIMENSIONAL GPS ATTITUDE DETERMINATION SYSTEM

The single difference carrier phase measurement is generally used in attitude determination using the GPS signal when a dedicated special purpose receiver is used [1,2]. To determine the 3-dimensional attitude, it is necessary to have  $m (\geq 4)$  visible GPS satellites and  $m_a (\geq 2)$  baselines or equivalently  $m_a+1$  antennas. The carrier phase measurement at the antenna  $i$  can be modeled as

$$\Phi_i - \lambda N_i = \rho_i + cB_i + w_i, \quad (1)$$

where  $\Phi_i = [\Phi_i^1 \ \dots \ \Phi_i^m]^T$ ,  $\rho_i = [\rho_i^1 \ \dots \ \rho_i^m]^T$ , and  $cB_i$  denote the carrier phase measurement vector, the distance vector between the antenna  $i$  and the  $m$  visible satellites, and the receiver clock bias vector, respectively,  $w_i = [w_i^1 \ \dots \ w_i^m]^T$  is the measurement noise with zero mean and variance of  $\sigma^2 I_m$ ,  $I_m$  is the  $m$  by  $m$  identity matrix and  $\sigma^2$  is the variance of the measurement noise. It is assumed that integer ambiguities  $N_i = [N_i^1 \ \dots \ N_i^m]^T$  have already been fixed using an adequate method [8,9]. In this paper, to simplify the analysis it is assumed that the multipath is minimized by appropriate methods such as choke ring, MEDLL, MET and so on [8]. Other common errors such as ionosphere and troposphere delay are ignored because the baseline length in the attitude determination system is usually very short, as shown in Fig. 1.

The linearized single difference observation equation at the reference antenna  $i$  and the antenna  $j$  can be written as

$$l_{ij} \equiv G_i r_j^e + w_{ij}, \quad (2)$$

where  $l_i = \Phi_i - \lambda N_i - \rho_{i0}$ ,  $l_j = \Phi_j - \lambda N_j - \rho_{j0}$ ,  $l_{ij}$

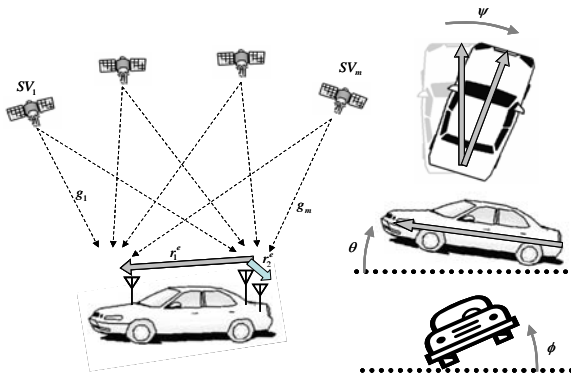


Fig. 1. The satellite-antenna geometry and the definition of attitude.

$= S [l_i^1 \ l_j^1 \ \dots \ l_i^m \ l_j^m]^T$ ,  $\rho_{i0}$  is a computed distance vector between the reference antenna  $i$  and the  $m$  visible satellites.  $G_i = [g_1 \ \dots \ g_m]^T$  is the matrix composed of the line of sight vectors  $g_k$  between the satellite  $k$  ( $=1, \dots, m$ ) and the antenna  $i$ ,  $r_j^e$  is the baseline vector from antenna  $i$  to  $j$ .

The single differenced measurement noise  $w_{ij} = S [w_i^1 \ w_j^1 \ \dots \ w_i^m \ w_j^m]^T$  has zero mean and variance of  $\sigma^2 SS^T$  where  $S$  is the single difference operator [4,9] defined by

$$S = \begin{bmatrix} 1 & -1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & -1 \end{bmatrix}. \quad (3)$$

Using the fact  $SS^T = 2I_m$ , the variance of  $w_{ij}$  becomes  $2\sigma^2 I_m$ .

Note that the receiver clock bias vector disappears in (2) by the single difference operation. It is assumed that both receivers  $i$  and  $j$  are driven by a common clock as in [1] and [2] and the line bias between the two receivers is removed.

The receiver clock bias vector can be removed when the double difference operation is taken to the carrier phase measurement [4,9]. In this case, it is possible to have an attitude determination system using low cost off-the-shelf GPS receivers. The single difference observation equation without the receiver clock bias is adopted in this paper in order to compare the result of [1] and [2].

The baseline vector  $r_j^e$  in the ECEF (Earth Centered Earth Fixed) coordinate frame can be estimated using the least squares method and is given by

$$r_j^e = (G_i^T G_i)^{-1} G_i^T l_{ij} \quad (4)$$

with the covariance matrix

$$\text{cov}(r_j^e) = 2\sigma^2 (G_i^T G_i)^{-1}. \quad (5)$$

The reference frame in which the attitude of the vehicle is specified is usually a local level frame such as the NED (North-East-Down) coordinate frame. Therefore, the baseline vector in the ECEF frame should be transformed to the NED frame. Using the matrix  $C_e^n$  of the coordinate transformation from the ECEF frame to the NED frame, the baseline vector  $r_j^n$  in the NED frame can be easily obtained as

$$r_j^n = C_e^n r_j^e, \quad (6)$$

whose covariance matrix is given by

$$Q_n = \text{cov}(r_j^n) = 2\sigma^2 C_n^e (G_i^T G_i)^{-1} C_n^e. \quad (7)$$

Denote  $r_j^b$  as the baseline vector between the reference antennas  $i$  and the antenna  $j$  in the body frame. Then  $r_j^b$  and  $r_j^n$  have the relation

$$R^b = C_n^b R^n, \quad (8)$$

$$C_n^b = \begin{bmatrix} c\theta c\psi & c\theta s\psi & -s\theta \\ -c\phi s\psi + s\phi s\theta c\psi & c\phi c\psi + s\phi s\theta s\psi & s\phi c\theta \\ s\phi s\psi + c\phi s\theta c\psi & -s\phi c\psi + c\phi s\theta s\psi & c\phi c\theta \end{bmatrix}, \quad (9)$$

where  $R^b = [r_1^b \ \dots \ r_{m_a}^b]$ ,  $R^n = [r_1^n \ \dots \ r_{m_a}^n]$ ,  $c$  and  $s$  mean  $\cos$  and  $\sin$  respectively.

$\phi$ ,  $\theta$ , and  $\psi$  are the roll, pitch and yaw, respectively. The roll axis points in the forward direction, the yaw axis points in the downward direction and the pitch axis completes the right-handed coordinate system.

As is well known, the coordinate transformation matrix  $C_n^b$  is orthogonal and since the positions of the antennas are known when installed,  $R^b$  is known *a priori*. Using (6),  $R^n$  is calculated from the measured vector,  $r_j^e$ ,  $j=1, \dots, m_a$ . Then  $C_n^b$  can be determined by applying the least squares method to (8) [2,10]. Therefore, it can be seen that the attitude error is represented by the measurement error in  $R^n$ .

Now, we will derive the attitude error equation. Let  $\phi = \phi_0 + \delta\phi$ ,  $\theta = \theta_0 + \delta\theta$ , and  $\psi = \psi_0 + \delta\psi$ . To a first-order approximation, we have

$$C_n^b = \bar{C}_n^b + \delta C_n^b, \quad (10) \quad \text{and}$$

$$\bar{C}_n^b = \begin{bmatrix} \cos \theta_0 \cos \psi_0 & \cos \theta_0 \sin \psi_0 & -\sin \theta_0 \\ -\cos \phi_0 \sin \psi_0 + \sin \phi_0 \sin \theta_0 \cos \psi_0 & \cos \phi_0 \cos \psi_0 + \sin \phi_0 \sin \theta_0 \sin \psi_0 & \sin \phi_0 \cos \theta_0 \\ \sin \phi_0 \sin \psi_0 + \cos \phi_0 \sin \theta_0 \cos \psi_0 & -\sin \phi_0 \cos \psi_0 + \cos \phi_0 \sin \theta_0 \sin \psi_0 & \cos \phi_0 \cos \theta_0 \end{bmatrix}, \quad (11)$$

$$\delta C_n^b = \begin{bmatrix} -(\sin \theta_0 \cos \psi_0) \delta\theta - (\cos \theta_0 \sin \psi_0) \delta\psi & -(\sin \theta_0 \sin \psi_0) \delta\theta + (\cos \theta_0 \cos \psi_0) \delta\psi & -(\cos \theta_0) \delta\theta \\ (\cos \phi_0 \sin \theta_0 \cos \psi_0 + \sin \phi_0 \sin \psi_0) \delta\phi & (\cos \phi_0 \sin \theta_0 \sin \psi_0 - \sin \phi_0 \cos \psi_0) \delta\phi & (\cos \phi_0 \cos \theta_0) \delta\phi \\ +(\sin \phi_0 \cos \theta_0 \cos \psi_0) \delta\theta & +(\sin \phi_0 \cos \theta_0 \sin \psi_0) \delta\theta & -(\sin \phi_0 \sin \theta_0) \delta\theta \\ -(\cos \phi_0 \cos \psi_0 + \sin \phi_0 \sin \theta_0 \sin \psi_0) \delta\psi & +(\sin \phi_0 \sin \theta_0 \cos \psi_0 - \cos \phi_0 \sin \psi_0) \delta\psi & -(\sin \phi_0 \cos \theta_0) \delta\phi \\ (\cos \phi_0 \sin \psi_0 - \sin \phi_0 \sin \theta_0 \cos \psi_0) \delta\phi & -(\cos \phi_0 \cos \psi_0 + \sin \phi_0 \sin \theta_0 \sin \psi_0) \delta\phi & -(\sin \phi_0 \cos \theta_0) \delta\phi \\ +(\cos \phi_0 \cos \theta_0 \cos \psi_0) \delta\theta & +(\cos \phi_0 \cos \theta_0 \sin \psi_0) \delta\theta & -(\cos \phi_0 \sin \theta_0) \delta\theta \\ +(\sin \phi_0 \cos \psi_0 - \cos \phi_0 \sin \theta_0 \sin \psi_0) \delta\psi & +(\cos \phi_0 \sin \theta_0 \cos \psi_0 + \sin \phi_0 \sin \psi_0) \delta\psi & \end{bmatrix}, \quad (12)$$

$$\Theta^\times = \bar{C}_n^b \delta C_n^b = \begin{bmatrix} 0 & -\sin \theta_0 \delta\phi + \delta\psi & -\cos \theta_0 \sin \psi_0 \delta\phi - \cos \psi_0 \delta\theta \\ \sin \theta_0 \delta\phi - \delta\psi & 0 & \cos \theta_0 \cos \psi_0 \delta\phi - \sin \psi_0 \delta\theta \\ \cos \theta_0 \sin \psi_0 \delta\phi + \cos \psi_0 \delta\theta & -\cos \theta_0 \cos \psi_0 \delta\phi + \sin \psi_0 \delta\theta & 0 \end{bmatrix}, \quad (13)$$

where  $\bar{C}_n^b$  and  $\delta C_n^b$  are in (11) and (12) respectively. From (11) and (12), we obtain the skew symmetric matrix in (13).

It is noted that  $R^n$  can be represented as

$$R^n = \bar{R}^n + \delta R^n, \quad (14)$$

where  $\bar{R}^n$  is the value that gives the true attitude  $\phi_0$ ,  $\theta_0$ , and  $\psi_0$ , i.e.,  $R^b = \bar{C}_n^b \bar{R}^n$ ;  $\delta R^n$  is the error resulting from the measurement noise. Using (8), (10) and the relation  $R^b = \bar{C}_n^b \bar{R}^n$ , it can be easily obtained that

$$\bar{C}_n^b \delta R^n + \delta C_n^b \bar{R}^n = 0. \quad (15)$$

By multiplying  $\bar{C}_n^b$  to both sides of (15) and rearranging them, we obtain

$$\delta R^n = -\bar{C}_n^b \delta C_n^b \bar{R}^n = -\Theta^\times \bar{R}^n. \quad (16)$$

The  $j$ -th column vector of  $\delta R^n$  is given by

$$\delta r_j^n = -\Theta^\times \bar{r}_j^n = (\bar{r}_j^n)^\times \delta \eta, \quad (17)$$

where  $(\bar{r}_j^n)^\times$  is the skew symmetric matrix of the baseline vector  $\bar{r}_j^n = [\bar{r}_{Nj}^n \ \bar{r}_{Ej}^n \ \bar{r}_{Dj}^n]^T$  and given by

$$(\bar{r}_j^n)^\times = \begin{bmatrix} 0 & -\bar{r}_{Dj}^n & \bar{r}_{Ej}^n \\ \bar{r}_{Dj}^n & 0 & -\bar{r}_{Nj}^n \\ -\bar{r}_{Ej}^n & \bar{r}_{Nj}^n & 0 \end{bmatrix} \quad (18)$$

$$\begin{aligned} \delta\eta &= \begin{bmatrix} -\cos\theta_0 \cos\psi_0 \delta\phi + \sin\psi_0 \delta\theta \\ -\cos\theta_0 \sin\psi_0 \delta\phi - \cos\psi_0 \delta\theta \\ \sin\theta_0 \delta\phi - \delta\psi \end{bmatrix} \\ &= \begin{bmatrix} -\cos\theta_0 \cos\psi_0 & \sin\psi_0 & 0 \\ -\cos\theta_0 \sin\psi_0 & -\cos\psi_0 & 0 \\ \sin\theta_0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \delta\phi \\ \delta\theta \\ \delta\psi \end{bmatrix} \equiv T \delta\xi. \end{aligned} \quad (19)$$

Using (17) and (19), we obtain

$$\begin{bmatrix} \delta r_1^n \\ \vdots \\ \delta r_{m_a}^n \end{bmatrix} = \begin{bmatrix} (\bar{r}_1^n)^\times \\ \vdots \\ (\bar{r}_{m_a}^n)^\times \end{bmatrix} T \delta\xi. \quad (20)$$

Applying the weighted least squares method [11], the attitude error can be estimated as

$$\delta\hat{\xi} = T^{-1} \left[ \sum_{j=1}^{m_a} ((\bar{r}_j^n)^\times Q_n^{-1} (\bar{r}_j^n)^\times) \right]^{-1} \left[ \sum_{j=1}^{m_a} ((\bar{r}_j^n)^\times Q_n^{-1} \delta r_j^n) \right] \quad (21)$$

with the covariance matrix

$$\text{cov}(\delta\hat{\xi}) = T^{-1} \left[ -\sum_{j=1}^{m_a} ((\bar{r}_j^n)^\times Q_n^{-1} (\bar{r}_j^n)^\times) \right]^{-1} T^{-T}. \quad (22)$$

From (7), it is unquestionable that all components of  $r_j^n$  are correlated with each other. For simplicity, however, we will follow the common assumption that all the components of  $r_j^n$  are uncorrelated with each other and have the same variance [1,3], that is,

$$Q_n = 2\sigma^2 I_3, \quad (23)$$

where  $I_3$  is the 3 by 3 identity matrix and the variance  $\sigma^2$  of an undifferenced carrier phase measurement is amplified to  $2\sigma^2$  by the single difference operation. Then, (22) becomes

$$\begin{aligned} \text{cov}(\delta\hat{\xi}) &= 2\sigma^2 T^{-1} \left[ -\sum_{j=1}^{m_a} ((\bar{r}_j^n)^\times (\bar{r}_j^n)^\times) \right]^{-1} T^{-T} \\ &= 2\sigma^2 T^{-1} \left[ \sum_{j=1}^{m_a} ((\bar{r}_j^n)^T \bar{r}_j^n I_3 - \bar{r}_j^n (\bar{r}_j^n)^T) \right]^{-1} T^{-T}. \end{aligned} \quad (24)$$

Because of assumption (23), the influence of both the geometry and the number of visible GPS satellites are not seen in (24). Since the length of the baseline vectors,  $b_j$ ,  $j=1, \dots, m_a$ , are constant under the coordinate transformation, (24) becomes

$$\begin{aligned} \text{cov}(\delta\hat{\xi}) &= 2\sigma^2 T^{-1} \left[ \sum_{j=1}^{m_a} (b_j^2 I_3 - \bar{r}_j^n (\bar{r}_j^n)^T) \right]^{-1} T^{-T} \\ &= 2\sigma^2 T^{-1} \left[ \sum_{j=1}^{m_a} (b_j^2 I_3 - \bar{C}_b^n r_j^b (r_j^b)^T \bar{C}_n^b) \right]^{-1} T^{-T}. \end{aligned} \quad (25)$$

The definition of the ADOP (Attitude DOP) in [2] and the EADOP in [1] can be derived from (25) by letting  $\phi_0 = \theta_0 = \psi_0 = 0$ , i.e.,  $T = -I_3$ . Equation (25) gives an explicit relationship among the attitude error, the baseline configuration, the nominal attitude of the vehicle and the receiver noise while the ADOP and the EADOP does not.

### 3. APPLICATIONS

Generally 2 or 3 baselines (3 or 4 antennas) are used in the GPS attitude determination application. The configurations of baselines are dependent on many factors like the size and shape of the vehicle. However, the 3 configurations represented in Fig. 2 are the most canonical forms when the use of the GPS attitude determination system is considered. Similarly, error analysis of another configuration can be easily done.

#### 3.1. Optimal 3-baselines configuration

Let the baseline vectors in the body frame be

$$r_1^b = b[1 \ 0 \ 0]^T, \quad r_2^b = b[0 \ 1 \ 0]^T, \quad r_3^b = b[0 \ 0 \ 1]^T. \quad (26)$$

Then, (25) becomes

$$\begin{aligned} \text{cov}(\delta\hat{\xi}) &= \frac{\sigma^2}{b^2} T^{-1} T^{-T} \\ &= \frac{\sigma^2}{b^2 \cos^2 \theta_0} \begin{bmatrix} 1 & 0 & \sin\theta_0 \\ 0 & \cos^2 \theta_0 & 0 \\ \sin\theta_0 & 0 & 1 \end{bmatrix}. \end{aligned} \quad (27)$$

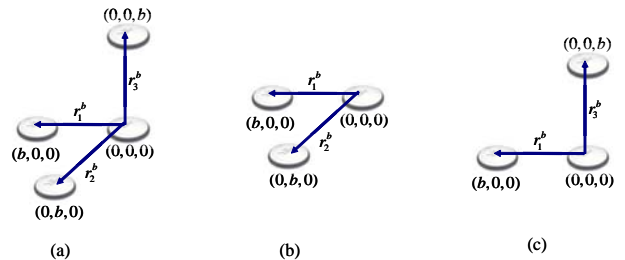


Fig. 2. Typical baseline configurations: (a) 3 orthogonal baselines with equivalent lengths, (b) 2 orthogonal baselines with equivalent lengths on a horizontal plane, (c) 2 orthogonal baselines with equivalent lengths on a vertical plane.

Equation (28) shows that the Euler angle errors can be reduced by increasing baseline length or reducing receiver noise. Furthermore, (27) implies that the pitch error is not affected by the nominal attitude of the vehicle while the roll and the yaw error increase as the vehicle inclines. The roll and the yaw error will diverge when  $\theta_0 = \pm 90^\circ$ , which requires caution in the airborne applications. This baseline configuration is known to be optimal because it has the minimum ADOP [3];

$$\begin{aligned} \text{ADOP} &\equiv \sqrt{\phi\text{DOP}^2 + \theta\text{DOP}^2 + \psi\text{DOP}^2} \\ &= \frac{1}{b \cos \theta_0} \sqrt{2 + \cos^2 \theta_0} \\ &= \frac{\sqrt{3}}{b}, \text{ if } \theta_0 = 0. \end{aligned} \quad (28)$$

### 3.2. Horizontal 2-baselines configuration

Consider the baseline configuration used in [1], that is,

$$r_1^b = b[1 \ 0 \ 0]^T, \quad r_2^b = b[0 \ 1 \ 0]^T. \quad (29)$$

Then, we have

$$\text{cov}(\delta \hat{\xi}) = \frac{\sigma^2}{b^2} T^{-1} (2 - C_b^n \mathbf{1}_3 \mathbf{1}_3^T C_n^b) T^{-T}, \quad (30)$$

where  $\mathbf{1}_3 = [0 \ 0 \ 1]^T$ . The EADOP can be obtained using

$$\text{diag}[\text{cov}(\delta \hat{\xi})] = \frac{\sigma^2}{b^2 \cos^2 \theta_0} \begin{bmatrix} 2 - \cos^2 \phi_0 \sin^2 \theta_0 \\ \cos^2 \theta_0 (2 - \sin^2 \phi_0) \\ 2 - \cos^2 \phi_0 \end{bmatrix}. \quad (31)$$

Equation (32) represents the affect of the nominal attitude of the vehicle, baseline length and receiver noise. For a stationary vehicle with  $\phi_0 = \theta_0 = \psi_0 = 0$ , EADOP becomes  $\phi\text{DOP} = \sqrt{2}\sigma/b$ ,  $\theta\text{DOP} = \sqrt{2}\sigma/b$  and  $\psi\text{DOP} = \sigma/b$ , which implies that the accuracy of the yaw angle is better than that of roll and pitch. This is coincident with the simulation results in [1]. However, it should be noted that the factors that influence the smaller  $\psi\text{DOP}$  than  $\phi\text{DOP}$  and  $\theta\text{DOP}$  are not only the satellites geometry but also the baseline configuration. Yoon and Lundberg's analysis which states that due to satellite geometry, HDOP is usually smaller than VDOP and that it gives smaller  $\psi\text{DOP}$ , is insufficient. In this configuration, two baselines are placed on a horizontal plane and one baseline is used to obtain roll and yaw angle while the other baseline is used to obtain pitch and yaw angle.

Therefore, the accuracy of yaw is superior to the roll and pitch with the factor of  $\sqrt{2}$ .

In this configuration, we have ADOP as

$$\begin{aligned} \text{ADOP} &= \frac{\sqrt{4 + \cos^2 \theta_0 (2 - \sin^2 \phi_0) - \cos^2 \phi_0 (\sin^2 \theta_0 + 1)}}{b \cos \theta_0} \\ &= \frac{\sqrt{5}}{b}, \text{ if } \theta_0 = \phi_0 = 0, \end{aligned} \quad (32)$$

which is larger than the optimal value in (28). It implies that the accuracy of attitude estimates improves with the number of baselines.

### 3.3. Vertical 2-baselines configuration

Consider the baseline configuration oriented in a north and downward direction

$$r_1^b = b[1 \ 0 \ 0]^T, \quad r_2^b = b[0 \ 0 \ 1]^T, \quad (33)$$

we have

$$\text{diag}[\text{cov}(\delta \hat{\xi})] = \frac{\sigma^2}{b^2 \cos^2 \theta_0} \begin{bmatrix} 2 - \sin^2 \phi_0 \sin^2 \theta_0 \\ \cos^2 \theta_0 (2 - \cos^2 \phi_0) \\ 2 - \sin^2 \phi_0 \end{bmatrix} \quad (34)$$

and

$$\begin{aligned} \text{ADOP} &= \frac{\sqrt{4 + \cos^2 \theta_0 (2 - \cos^2 \phi_0) - \sin^2 \phi_0 (\sin^2 \theta_0 + 1)}}{b \cos \theta_0} \\ &= \frac{\sqrt{5}}{b}, \text{ if } \theta_0 = \phi_0 = 0. \end{aligned} \quad (35)$$

Equation (35) represents the affect of the nominal attitude of the vehicle, baseline length and receiver noise. For a stationary vehicle with  $\phi_0 = \theta_0 = \psi_0 = 0$ , EADOP becomes  $\phi\text{DOP} = \sqrt{2}\sigma/b$ ,  $\theta\text{DOP} = \sigma/b$  and  $\psi\text{DOP} = \sqrt{2}\sigma/b$ , which implies that the accuracy of pitch angle is better than that of roll and yaw.

Equations (28), (32) and (35) clearly show the impacts of the antenna configuration and the nominal attitude of the vehicle on the attitude estimates. It can be easily applied to predict and analyze the attitude error. However, note that the influence of the geometry and the number of visible satellites is not included because of assumption (23). In order to obtain more rigorous results that consider all parameters, (22) should be used.

## 4. CONCLUDING REMARKS

Error analysis of 3 dimensional GPS attitude determination is given in this paper. An analytic

expression of the attitude error covariance is derived for the rigorous analysis. The new EADOP expressions, which can be a simple but efficient analysis tool for the attitude error analysis, are also derived. Using these expressions the error can be analyzed without computer simulations because they explicitly represent the attitude error as the function of the nominal attitude of the vehicle, the baseline configuration and the receiver noise. The applications to some baseline configurations demonstrate the effectiveness of the derived expressions. These results can be applied to predict and analyze the performance of the GNSS attitude determination system.

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