Time-Discretization of Time Delayed Non-Affine System via Taylor-Lie Series Using Scaling and Squaring Technique

Yuanliang Zhang and Kil To Chong*

Abstract: A new discretization method for calculating a sampled-data representation of a nonlinear continuous-time system is proposed. The proposed method is based on the well-known Taylor series expansion and zero-order hold (ZOH) assumption. The mathematical structure of the new discretization method is analyzed. On the basis of this structure, a sampled-data representation of a nonlinear system with a time-delayed input is derived. This method is applied to obtain a sampled-data representation of a non-affine nonlinear system, with a constant input time delay. In particular, the effect of the time discretization method on key properties of nonlinear control systems, such as equilibrium properties and asymptotic stability, is examined. 'Hybrid' discretization schemes that result from a combination of the 'scaling and squaring' technique with the Taylor method are also proposed, especially under conditions of very low sampling rates. Practical issues associated with the selection of the method parameters to meet CPU time and accuracy requirements are examined as well. The performance of the proposed method is evaluated using a nonlinear system with a time-delayed non-affine input.

Keywords: Non-affine, nonlinear system, scaling and squaring technique, stability, Taylor series, time delay, time discretization.

1. INTRODUCTION

Time-delay systems (TDS) are also referred to as systems with aftereffects or dead-time, hereditary systems, equations with a deviating argument, or differential-difference equations. The development and evolution of Internet technology has increased interest in control systems with time delays due to the convergence of communication and computations in control systems and the complex behavior of control systems with non-negligible time delays. Digital controllers using communications and increased computation requirements in systems induce the time delay. Also, time-delay systems often appear in industrial systems and information networks. Thus, it is important to analyze time-delay systems and design appropriate controllers. Control systems with time delays exhibit complex behaviors because of their infinite dimensionality. Even in the case of linear time-invariant systems that have constant time delays in their inputs or states have infinite dimensionality if expressed in the continuous time domain. It is therefore difficult to apply the controller design techniques that have been developed during the last several decades for finite-dimensional systems to systems with any time delays in the variables. Thus, new control system design methods that can solve a system with time delays are necessary.

Traditional numerical schemes for ordinary differential equations, such as Runge-Kutta schemes, usually fail to attain their asserted order when applied to ordinary differential control equations due to the measurability of the control functions. Grüne and Kloeden extended a systematic method for deriving high-order schemes for affine controlled nonlinear systems to a larger class of systems in which the control variables were allowed to appear nonlinearly in multiplicative terms [1].

In the field of discretization, traditional numerical techniques, such as the Euler and Runge-Kutta methods, have been used to obtain sampled-data representations of original continuous-time systems in the delay-free case [2]. But these methods require a small sampling time interval to meet the desired accuracy; they cannot be applied to cases with the large sampling periods. Due to the physical and technical limitations, slow sampling has become inevitable. A time discretization method that expands

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the well-known time-discretization of linear time-delay systems [2,3] to nonlinear continuous-time control systems with time delays [4-6] can solve this problem. The effect of this approach on system-theoretic properties of nonlinear systems, such as equilibrium properties, relative order, stability, zero dynamics, and minimum-phase characteristics has also been studied [7] and reveals the natural and transparent manner in which Taylor methods permeate the relevant theoretical aspects.

At present, modern nonlinear control strategies are usually implemented using a microcontroller or digital signal processor. As a direct consequence, control algorithms must operate using discrete time intervals. The time discretization is based either on a continuous-time control law designed assuming a continuous-time system, or on a discrete-time control law designed for a continuous-time system that results in a discrete-time system. It is apparent that the second approach is an attractive feature for dealing directly with the issue of sampling. The effect of sampling on the system-theoretic properties of a continuous-time system is very important because it is associated with attaining the design objectives. It should be emphasized that in both design approaches, a time discretization of either the controller or the system model is necessary. Furthermore, notice that in the controller design for time–delay systems, the first approach is troublesome because of the infinite-dimensional nature of the underlying system dynamics. As a result, the second approach becomes more desirable and will be pursued in the present study.

This paper proposes a time discretization method for nonlinear control systems with non-affine time-delay control inputs. The proposed discretization scheme applies a Taylor series expansion according to the mathematical structure developed for delay-free nonlinear systems [7,8]. The effect of sampling on the system-theoretic properties of nonlinear systems with time-delayed non-affine inputs, such as equilibrium properties and stability, is examined. Also, the well-known "scaling and squaring" technique (SST), which is widely used for computing the matrix exponential [9], is applied to the nonlinear case, when the sampling period is too large.

In particular, the paper makes the following contributions:

- A new method is proposed for the time-discretization of nonlinear dynamic systems with time-delay non-affine input based on Taylor-Lie series; the resulting descrete-time system (or sampled-data representation) preserves/inherits some of the system-theoretic properties of the original continuous-time system (such as equilibrium and stability properties), and also it is finite dimensional, thus allowing the direct application of existing nonlinear control system design techniques;
- The 'hybrid' discretization schemes, that results from a combination of the 'scaling and squaring' (or extrapolation to the limit) technique with the Taylor method is proposed when the sampling period is large.
- The proposed method and discretization algorithm is tested by using a nonlinear system with time-delay non-affine input. For this nonlinear control system various sampling rates and time-delay values are considered, demonstrating the accuracy of the proposed discretization scheme.

The paper is organized as follows: Section 2 first presents some mathematical preliminaries regarding the structure of the nonlinear time-delay control system investigated; briefly presents the time-discretization of nonlinear systems with delay free input; and part 2.2 contains the main results of this paper, where a new time-discretization method and procedure for nonlinear control systems with time-delay non-affine input is introduced. Section 3 presents the 'hybrid' discretization scheme that results from a combination of the 'scaling and squaring' technique with the Taylor method when the sampling period is large. Finally, Section 4 includes a nonlinear continuous system with non-affine input studies demonstrating the effectiveness of the proposed discretization scheme, whereas Section 5 provides a few concluding remarks drawn from this study.

2. TIME-DISCRETIZATION OF DELAYED NON-AFFINE NONLINEAR SYSTEMS

In the present study, single-input nonlinear continuous-time control systems are considered with state-space representations of the form [4]:

$$\frac{dx(t)}{dt} = f(x(t)) + g(x(t))u(t-D), \tag{1}$$

where $x \in X \subset R^n$ is the state vector, X is an open and connected set, $u \in R$ is the input variable, and D is the system constant time delay (dead-time) that directly affects the input. It is assumed that $f(x)$, $g(x)$ are real analytic vector fields on X.

An equidistant grid on the time axis with a mesh $T = t_{k+1} - t_k > 0$ is considered, where $[t_k, t_{k+1}) = [kT, (k+1)T)$ is the sampling interval and T is the sampling period. It is assumed that system (1) is driven by an input that is piecewise constant over the sampling interval, i.e. the zero-order hold (ZOH) assumption holds true:

$$u(t) = u(kT) \equiv u(k) = \text{constant}, \tag{2}$$

for $kT \le t < kT + T$. Furthermore, let:

$$D = qT + \gamma \,, \tag{3}$$

where $q \in \{0,\ 1,\ 2,\ \dots\}$ and $0 < \gamma < T$, i.e. the time-delay D can be represented as an integer multiple of the sampling period plus a fractional part of T [2,3].

2.1. Discretization of nonlinear systems with delay-free inputs

Delay-free $(D = 0)$ nonlinear control systems are considered with state-space representations of the form:

$$\frac{dx(t)}{dt} = f(x(t)) + u(t)g(x(t)) \,. \tag{4}$$

Within the sampling interval and under the ZOH assumption, the solution of (4) can be expanded in a uniformly convergent Taylor series [10]:

$$\begin{aligned} x(k+1) &= \Phi_T(x(k), u(k)) \\ &= x(k) + \sum_{l=1}^{\infty} \frac{T^l}{l!} \frac{d^l x}{dt^l}\bigg|_{t_k} \\ &= x(k) + \sum_{l=1}^{\infty} A^{[l]}(x(k), u(k)) \frac{T^l}{l!}, \end{aligned} \tag{5}$$

where $x(k)$ is the value of the state vector x at time $t = t_k = kT$ and $A^{[l]}(x,u)$ are determined recursively by

$$\begin{aligned} &A^{[l]}(x,u) = f(x) + ug(x) \,, \\ &A^{[l+1]}(x,u) = \frac{\partial A^{[l]}(x,u)}{\partial x}(f(x) + ug(x)) \\ &\qquad \text{with } l = 1,\ 2,\ 3,\ \dots\ . \end{aligned} \tag{6}$$

2.2. Discretization of nonlinear systems with delayed non-affine input

The equations of time-delayed single non-affine input nonlinear control systems are as follows [11,12];

$$\begin{aligned} \dot{x}(t) = {} & f_0(x(t)) + g_1(x(t))u(t-D) \\ & + g_2(x(t))u(t-D)^2 + \dots + g_m(x(t))u(t-D)^m, \end{aligned}$$

$$\text{and} \quad \dot{x}(t) = f(x(t), u(t-D)) \,, \tag{7}$$

where $x \in R^n$ is the state, $u \in R$ is the control input, $f_0 : R^n \to R^n$, $g_i : R^n \to R^n$, $i = 1, 2, \dots, m$ and $f : R^n \times R \to R^n$ are smooth mappings.

Based on the ZOH assumption and the above notation one can deduce that the delayed input variable attains the following two distinct values within the sampling interval:

$$u(t-D) = \begin{cases} u(kT - qT - T) \equiv u(k-q-1) \\ \qquad \text{if } kT \le t < kT + \gamma \\ u(kT - qT) \equiv u(k-q) \\ \qquad \text{if } kT + \gamma \le t < kT + T. \end{cases} \tag{8}$$

Non-affine nonlinear systems can also be discretized using Taylor series expansions. From the ZOH assumption, we obtain;

- $kT \le t < kT + \gamma$

  The exact sampled-data representation *(ESDR)* is

$$\begin{aligned} x(kT+\gamma) &= x(kT) \\ &+ \sum_{l=1}^{\infty} A^l(x(kT),\ u(k-q-1)) \frac{\gamma^l}{l!}, \end{aligned} \tag{9}$$

and the approximate sampled-data representation *(ASDR)* resulting from truncating the Taylor series at order N is

$$\begin{aligned} x(kT+\gamma) &= x(kT) \\ &+ \sum_{l=1}^{N} A^l(x(kT), u(k-q-1)) \frac{\gamma^l}{l!}, \end{aligned} \tag{10}$$

- $kT + \gamma \le t < kT + T$

  The ESDR is

$$\begin{aligned} x(kT+T) &= x(kT+\gamma) \\ &+ \sum_{l=1}^{\infty} A^l(x(kT+\gamma), u(k-q)) \frac{(T-\gamma)^l}{l!}, \end{aligned} \tag{11}$$

and the ASDR is

$$\begin{aligned} x(kT+T) &= x(kT+\gamma) \\ &+ \sum_{l=1}^{N} A^l(x(kT+\gamma), u(k-q)) \frac{(T-\gamma)^l}{l!}. \end{aligned} \tag{12}$$

The generalized coefficients are represented as follows:

$$\begin{aligned} &A^{[1]}(x,u) = f(x,u),\ A^{[l+1]}(x,u) \\ &\qquad = \frac{\partial A^{[l]}(x,u)}{\partial x} f(x,u). \end{aligned} \tag{13}$$

Definition 1: Given f, an analytic vector field on R^n and h, an analytic scalar field on R^n, the Lie derivative of h with respect to f is defined in local coordinates as

$$L_f h(x) = \frac{\partial h}{\partial x_1} f_1 + \dots + \frac{\partial h}{\partial x_n} f_n \,. \tag{14}$$

In light of Definition 1, the solution to the recursive relation (13) may be represented in terms of higher-order Lie derivatives as follows:

$$A^{[l]}(x,u)=(L_{f_0}+uL_{g_1}+u^2L_{g_2}+...+u^mL_{g_m})^l x, \quad (15)$$

where $L_{f_0}=f_0(x)\frac{\partial}{\partial x}$, $L_{g_1}=g_1(x)\frac{\partial}{\partial x}$, $L_{g_2}=g_2(x)\frac{\partial}{\partial x}$, ... , $L_{g_m}=g_m(x)\frac{\partial}{\partial x}$ are Lie derivative operators.

Theorem 1: Let x^0 be an equilibrium point of the original non-affine input nonlinear continuous system

$$\dot{x}(t)=f_0(x)+ug_1(x)+u^2g_2(x)+...+u^mg_m(x), \quad (16)$$

that belongs to the equilibrium manifold

$$E^c=\left\{x\in R^n \middle| \exists u\in R: f(x,u)=0\right\}. \quad (17)$$

Let $u=u^0$ be the corresponding equilibrium value of the input variables $f(x^0,u^0)=0$. Then x^0 belongs to the equilibrium manifold: $E^d=\left\{x\in R^n \middle| \exists u\in R: \Phi_T^D(x,u)=x\right\}$ of the ESDR: $x(k+1)=\Phi_T^D(x(k), u(k-q-1), u(k-q))$ and ASDR: $x(k+1)=\Phi_T^{N,D}(x(k), u(k-q-1), u(k-q))$, where is obtained using the proposed Taylor-Lie discretization method, with $u=u^0$ being the corresponding equilibrium values of the input variables: $\Phi_T^D(x^0,u^0)=x^0$ and $\Phi_T^{N,D}(x^0,u^0)=x^0$.

Proof: x^0 is the equilibrium point and u^0 are the corresponding equilibrium values of the input variables $\Longrightarrow A^{[1]}(x^0,u^0)=f_0(x^0)+u^0g_1(x^0)+(u^0)^2 g_2(x^0)+ ... +(u^0)^m g_m(x^0)=0 \Longrightarrow A^{[l+1]}(x^0,u^0)=\frac{\partial A^{[l]}(x^0,u^0)}{\partial x}A^{[1]}(x^0,u^0)=0,$ for all $l\in\{1,2,3,...\}$.

In the time interval $t\in[kT,kT+\gamma]$

$$\Longrightarrow \Phi_\gamma(x^0,u^0)=x^0+\sum_{l=1}^{\infty}A^{[l]}(x^0,u^0)\frac{\gamma^l}{l!}=x^0.$$

In the time interval $t\in[kT+\gamma,kT+T]$

$$\Phi_T^D(x^0,u^0)=\Phi_{T-\gamma}(\Phi_\gamma(x^0,u^0),u^0)=x^0.$$

Similar arguments apply to the $\Phi_T^{N,D}$ map of the ASDR. Therefore, x^0 belongs to the equilibrium manifold E^d of the ESDR and ASDR for any finite truncation order N.

The following technical lemma is essential [7].

Lemma 1: In the single input status, let x^0 be an equilibrium point of $\dot{x}(t)=f(x(t))+g(x(t))u(t-D)$ that corresponds to $u=u^0$. For any analytic scalar field $h(x)$, and positive integer l, the following equality holds:

$$\frac{\partial}{\partial x}\left[L_f+uL_g\right]^l h(x)\Big|_{(x^0,u^0)}=\frac{\partial h}{\partial x}\left[\frac{\partial f}{\partial x}+u^0\frac{\partial g}{\partial x}\right]^l(x^0). \quad (18)$$

The $i-th$ row of matrix $\frac{\partial\Phi_T^N}{\partial x}(x^0,u^0)$ can be calculated as follows:

$$\begin{aligned}\frac{\partial\Phi_{i,T}^N}{\partial x}(x^0,u^0)&=\sum_{l=0}^{N}\frac{\partial}{\partial x}\Big[(L_f+uL_g)^l x_i\Big]\Big|_{(x^0,u^0)}\frac{T^l}{l!}\\&=\sum_{l=0}^{N}\frac{\partial x_i}{\partial x}\Big(\frac{\partial f}{\partial x}+u^0\frac{\partial g}{\partial x}\Big)^l(x^0)\frac{T^l}{l!}.\end{aligned} \quad (19)$$

Theorem 2: Assume that matrix $M=\left[\frac{\partial f_0}{\partial x}+u^0\frac{\partial g_1}{\partial x}+(u^0)^2\frac{\partial g_2}{\partial x}+...+(u^0)^m\frac{\partial g_m}{\partial x}\right](x^0)$ is a Hurwitz matrix, so that x^0 is a locally asymptotically stable equilibrium point of the delay-free system:

$$\begin{aligned}\dot{x}(t)=\ &f_0(x(t))+g_1(x(t))u(t)\\&+g_2(x(t))u^2(t)+...+g_m(x(t))u^m(t).\end{aligned} \quad (20)$$

Then x^0 is a locally asymptotically stable equilibrium point of the ESDR and ASDR for sufficiently large N when T is fixed.

Proof: From Lemma 1,

$$\begin{aligned}\frac{\partial\Phi_T^N}{\partial x}(x^0,u^0)&=\sum_{l=0}^{N}\begin{pmatrix}\Big(\frac{\partial f_0}{\partial x}+u^0\frac{\partial g_1}{\partial x}+(u^0)^2\frac{\partial g_2}{\partial x}+...\\+(u^0)^m\frac{\partial g_m}{\partial x}\Big)^l(x^0)\frac{T^l}{l!}\end{pmatrix}\\&=\sum_{l=0}^{N}M^l\frac{T^l}{l!}.\end{aligned} \quad (21)$$

For an ASDR of finite truncation order *N*, or for the ESDR $(N\to\infty)$:

$$\begin{aligned}\frac{\partial\Phi_T}{\partial x}(x^0,u^0)&=\exp\begin{bmatrix}\Big(\frac{\partial f_0}{\partial x}+u^0\frac{\partial g_1}{\partial x}+(u^0)^2\frac{\partial g_2}{\partial x}+\\...+(u^0)^m\frac{\partial g_m}{\partial x}\Big)(x^0)T\end{bmatrix}\\&=\exp(MT).\end{aligned} \quad (22)$$

i) Consider now the ESDR with time-delay *D*. Notice that:

$$\begin{aligned}\frac{\partial \Phi_T^D}{\partial x}(x^0,u^0) &= \frac{\partial \Phi_{T-\gamma}}{\partial x}(\Phi_\gamma(x^0,u^0),u^0)\frac{\partial \Phi_\gamma}{\partial x}(x^0,u^0)\\ &= \frac{\partial \Phi_{T-\gamma}}{\partial x}(x^0,u^0)\frac{\partial \Phi_\gamma}{\partial x}(x^0,u^0) \qquad (23)\\ &= \exp(M(T-\gamma))\exp(M\gamma) = \exp(MT).\end{aligned}$$

Since M is a Hurwitz matrix, it can be inferred that all the eigenvalues of $\frac{\partial \Phi_T^D}{\partial x}(x^0,u^0)$ have a modulus of less than 1. Hence x^0 is a locally asymptotically stable equilibrium point of the ESDR.

ii) Consider now the ASDR. Here,

$$\begin{aligned}\frac{\partial \Phi_T^{N,D}}{\partial x}(x^0,u^0) &= \frac{\partial \Phi_{T-\gamma}}{\partial x}(\Phi_\gamma(x^0,u^0),u^0)\frac{\partial \Phi_\gamma}{\partial x}(x^0,u^0)\\ &= \frac{\partial \Phi_{T-\gamma}}{\partial x}(x^0,u^0)\frac{\partial \Phi_\gamma}{\partial x}(x^0,u^0)\\ &= \{\sum_{l_2=0}^{N} M^{l_2}\frac{(T-\gamma)^{l_2}}{l_2!}\}\{\sum_{l_1=0}^{N} M^{l_1}\frac{(\gamma)^{l_1}}{l_1!}\}\\ &= \sum_{l_2=0}^{N}\sum_{l_1=0}^{N} M^{l_2+l_1}\frac{(T-\gamma)^{l_2}(\gamma)^{l_1}}{l_2!\,l_1!}.\end{aligned} \qquad (24)$$

Notice that for a stable eigenvalue λ_i of M $(\mathrm{Re}[\lambda_i]<0)$, the corresponding eigenvalue a_i of $\frac{\partial \Phi_T^{N,D}}{\partial x}(x^0,u^0)$: $a_i = \sum_{l_2=0}^{N}\sum_{l_1=0}^{N}\lambda_i^{l_2+l_1}\frac{(T-\gamma)^{l_2}(\gamma)^{l_1}}{l_2!\,l_1!}$ is stable only when $|a_i|<1$.

Since for a fixed T and as $N\to\infty, a_i\to\exp(\lambda_i(T-\gamma))\exp(\lambda_i\gamma)=\exp(\lambda_i T)$, one can always find a sufficiently large order of truncation N such that: $|a_i|<1$. Therefore x^0 is a locally asymptotically stable equilibrium point of the ESDR and the ASDR for sufficiently large N when T is fixed.

3. SCALING AND SQUARING TECHNIQUE (SST)

The Taylor-Lie series method gives the necessarily accurate results, provided that the order N is very large if the sampling interval T is large. But when T is large, $A^{[l]}T^l/l!$ becomes extremely large due to the finite-precision arithmetic before it becomes small at higher powers when convergence takes over. In the case of a linear system, this phenomenon occurs when calculating e^{AT} and $\int_0^T e^{AT}dt$, which causes overflow errors in the computational number representations.

A scaling and squaring technique which is also called an extrapolation to the limit technique in numerical analysis literature can be applied to solve this type of problem. This technique is commonly used to calculate the exponential matrix exp(AT) for large sampling periods. By applying the SST, one can subdivide the sampling interval T into two or more subintervals of equal length. An appropriate positive integer m can be chosen such that $T/2^m$ is sufficiently small to calculate the exponential matrix. In this case, the sampling period T is subdivided into 2^m equally spaced subintervals of length $T/2^m$ and the exponential matrix is calculated for the short interval $T/2^m$. Finally, the computation of exp(AT) is performed by squaring the matrix $\exp(AT/2^m)$ m times:

$$\exp(AT) = (((\exp(A\frac{T}{2^m}))^2)\cdots)^2\,. \qquad (25)$$

The SST can be extended to nonlinear cases by applying the Taylor-Lie series method. Analogous to the linear case, one can use nonlinear operators and powers of operators as substitutes for matrices and matrix products. The key idea of the nonlinear application of the SST remains the same as for the linear case.

In the nonlinear case when T is sufficiently large, one can divide the interval $[t_k, t_{k+1})$ to 2^m equally spaced subintervals and use a small Taylor expansion order N with a time step $T/2^m$ for the 2^m intermediate subintervals to substitute the larger order N' used in the single-step Taylor method case.

Assume now that $\Omega(N',T): R^n \to R^n$ is the operator that corresponds to the Taylor expansion of order N' with a time step T, and when it acts on $x(kT)$ the outcome is:

$$x(kT+T) = \Omega(N',T)x(kT)\,, \qquad (26)$$

where

$$\Omega(N',T)(\cdot) = I + \sum_{l=1}^{N'} A^{[l]}(x(k),u(k))\frac{T^l}{l!}\,. \qquad (27)$$

Using operator notation the resulting discrete-time system may now be written as follows:

$$x(kT+T) = \left[\Omega\left(N,\frac{T}{2^m}\right)\right]^{2^m} x(kT)\,. \qquad (28)$$

The above ASDR may be viewed as the direct result of the combination of Taylor's method and the SST.

The choice of the parameters of N and m is an important consideration. Different values of N and m can reflect different requirements of the discretization performance. In this paper, we focused on simplicity and computing time, and numerical convergence and accuracy requirements to select these parameters. The criterion for selecting an appropriate m is a comparison of the magnitude of the sampling period T with the fastest time constant $1/\rho$ of the original continuous-time system. If T is small compared to $2/\rho$, we can set $m=0$ and we apply the single-step Taylor-Lie series method. Since T is small, a low order N single-step Taylor discretization method is usually sufficient to meet the expected accuracy requirements. If T is larger than the fastest time constant $2/\rho$, we must apply the SST. The sampling interval is therefore subdivided into 2^m subintervals, and a low-order N single step Taylor discretization method is used for each subinterval. This requests that the following inequality holds:

$$\frac{T}{2^m} < \frac{2}{\rho}. \tag{29}$$

Since the requirements for numerical convergence and stability are also met, the positive integer m is now selected to be

$$m = \max\left(\left[\log_2\left(\frac{T}{\theta}\right)\right]+1, 0\right), \tag{30}$$

where $\theta < 2/\rho$ is chosen arbitrarily and $[x]$ denotes the integer part of the number x. It is evident that smaller values of the arbitrarily selected number θ would result in more stringent bounds on $T/2^m$.

The SST can also be applied to nonlinear control systems with time-delayed non-affine inputs. In this case, we do not consider the single sampling interval T but only the subintervals of $\gamma, T-\gamma$. The same method can be used to select m by changing T of the preceding equality into the subintervals of $\gamma, T-\gamma$, i.e.,

$$m_\gamma = \max\left(\left[\log_2\left(\frac{\gamma}{\theta}\right)\right]+1, 0\right), \tag{31}$$

$$m_{T-\gamma} = \max\left(\left[\log_2\left(\frac{T-\gamma}{\theta}\right)\right]+1, 0\right). \tag{32}$$

4. SIMULATIONS

The performance of the proposed time discretization of non-affine nonlinear systems with input time delays using the Taylor series expansion method with the SST was evaluated by applying it to a non-affine system. This system exhibited nonlinear behavior and was studied for a broad range of sampling periods and input delay values. In this paper, the continuous Matlab ODE solver was used to obtain an exact solution for the system in order to validate the proposed discretization method.

The system considered in this paper was a single-input non-affine nonlinear system with a small fastest time constant $1/\rho$.

$$\begin{aligned} \dot{x}_1 &= -x_1^3 + x_1 x_2 u^2 + 3\cos(x_2)x_1, \\ \dot{x}_2 &= x_2^2 u \end{aligned} \tag{33}$$

The largest eigenvalue of the linear approximation of this system was 3, so that $2/\rho = 0.6667$. The input u was assumed to be $u = 3 + 0.4\cos(0.4x_2)$. The stability of the system changed with the initial conditions, which are assumed to be $x_1(0)=1$, $x_2(0)=-1$ in order to make the system stable. First, we chose a small sampling period and a small time delay to validate the discretization method proposed in this paper. The sampling period (T) is 0.02sec and the input delay is 0.028sec. In this case, $T < 2/\rho$, and therefore, a low order N single-step Taylor method was appropriate. Indeed, even a third-order $N=3$ single-step Taylor method provided an accurate discrete-time model. The resulting value of states x_1 and x_2 which are responses of the Taylor method and the Matlab solutions are shown in Table 1. The results obtained from these two methods are also shown in Fig. 1. Differences between the responses of the two methods are illustrated in Fig. 2. The results demonstrated that the proposed discrtzation method for nonlinear systems with delayed non-affine inputs was sufficiently accurate.

We then chose a sampling period of $T = 0.10s$ and an input delay of $D = 0.06s$. First we used the single-step Taylor method with $N=3$. The results of states x_1 and x_2 which are response of the Taylor method and Matlab solutions, states x_1 and x_2, are

Table 1. State response of the system for case 1.

Time step	State x1		State x2	
	Matlab	Maple (N=3)	Matlab	Maple (N=3)
100	0.2341	0.2350	-0.1299	-0.1300
200	1.4509	1.4509	-0.0690	-0.0690
300	1.5602	1.5602	-0.0470	-0.0470
400	1.6038	1.6051	-0.0356	-0.0356
500	1.6312	1.6313	-0.0287	-0.0287
600	1.6483	1.6485	-0.0240	-0.0240
700	1.6606	1.6607	-0.0206	-0.0206
800	1.6696	1.6698	-0.0181	-0.0181
900	1.6758	1.6768	-0.0161	-0.0161
1000	1.6819	1.6824	-0.0145	-0.0145

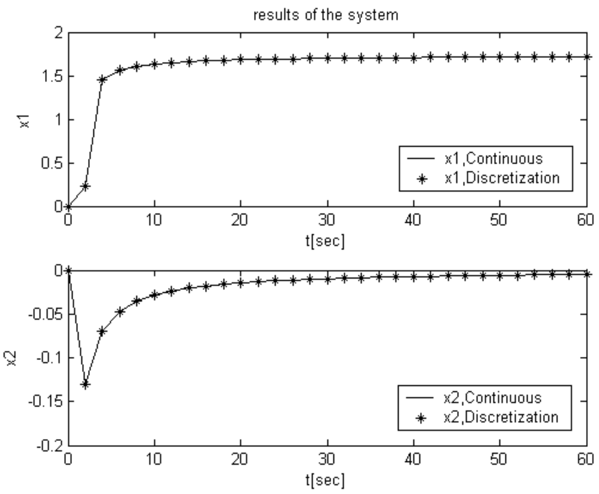

Fig. 1. State results of the system for case 1.

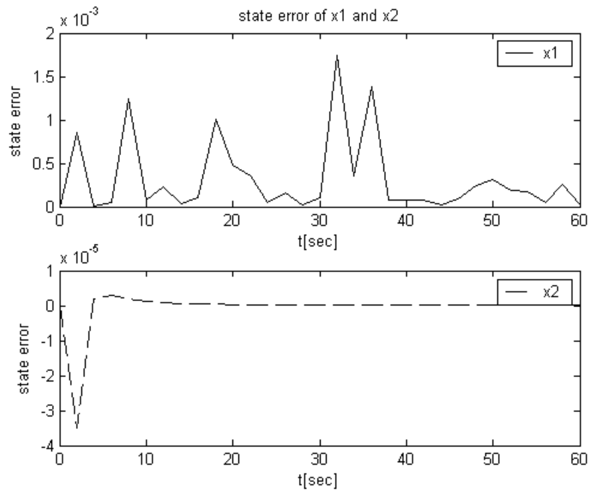

Fig. 2. State error response of the system for case 1.

Table 2. State response of the system for case 2(N=3).

Time Step	State x1		State x2	
	Matlab	Maple	Matlab	Maple
20	0.2272	0.2628	-0.1318	-0.1324
40	1.4470	1.4495	-0.0695	-0.0697
60	1.5592	1.5589	-0.0472	-0.0472
80	1.6037	1.6044	-0.0357	-0.0358
100	1.6309	1.6308	-0.0287	-0.0288
120	1.6480	1.6482	-0.0240	-0.0241
140	1.6603	1.6605	-0.0207	-0.0207
160	1.6695	1.6696	-0.0181	-0.0181
180	1.6751	1.6766	-0.0161	-0.0161
200	1.6822	1.6823	-0.0145	-0.0145

shown in Table 2. Then we used $N=7$; the resulting states x_1 and x_2 which are response of the Taylor method and Matlab solution are shown in Table 3. Differences in response for these two cases with the Matlab solution are shown in Fig. 3. The computing

Table 3. State response of the system for case 2(N=7).

Time Step	State x1		State x2	
	Matlab	Maple	Matlab	Maple
20	0.2272	0.2278	-0.1318	-0.1318
40	1.4470	1.4470	-0.0695	-0.0695
60	1.5592	1.5592	-0.0472	-0.0472
80	1.6037	1.6046	-0.0357	-0.0357
100	1.6309	1.6310	-0.0287	-0.0287
120	1.6480	1.6483	-0.0240	-0.0240
140	1.6603	1.6605	-0.0207	-0.0207
160	1.6695	1.6696	-0.0181	-0.0181
180	1.6751	1.6767	-0.0161	-0.0161
200	1.6822	1.6823	-0.0145	-0.0145

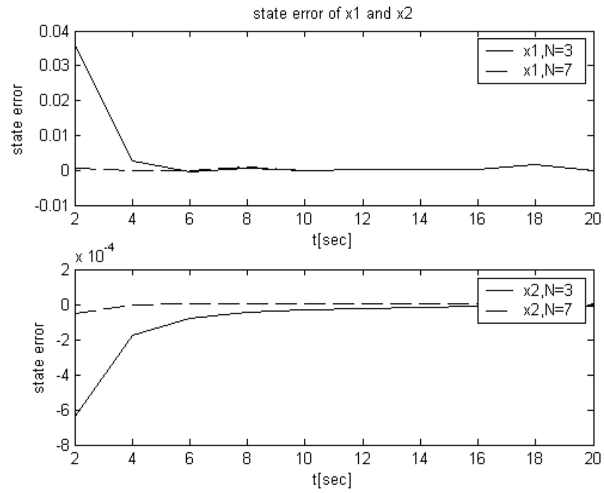

Fig. 3. State error response of the system for case 2.

Table 4. State response of the system for case 3.

Time step	State x1			
	Matlab	Maple		
		N=2	N=7	N=10
2	0.0979	*	*	*
4	0.3294	*	*	*
6	1.2847	*	*	*
8	1.5007	*	*	*
10	1.5531	*	*	*
Time step	State x2			
	Matlab	Maple		
		N=2	N=7	N=10
2	-0.2368	-2.0420	*	*
4	-0.1205	*	*	*
6	-0.0808	*	*	*
8	-0.0608	*	*	*
10	-0.0487	*	*	*

*Denotes order of magnitude greater than 10^5.

time used by these two cases were 0.28s and 17.76s, respectively. These results indicate that the Taylor series expansion order N must be enlarged when the sampling interval is large to improve the accuracy of

Table 5. State response of the system for Case 3 (SST).

Time step	State x1 (Maple)		
	N=3 l=4,m=4	N=7 l=5,m=5	N=3 l=10,m=10
2	0.1005	0.1009	0.1009
4	0.3357	0.3368	0.3368
6	1.2873	1.2880	1.2880
8	1.5006	1.5006	1.5006
10	1.5530	1.5530	1.5530
Time step	State x2 (Maple)		
	N=3 l=4,m=4	N=7 l=5,m=5	N=3 l=10,m=10
2	-0.2376	-0.2376	-0.2376
4	-0.1207	-0.1207	-0.1207
6	-0.0809	-0.0809	-0.0809
8	-0.0608	-0.0608	-0.0608
10	-0.0487	-0.0487	-0.0487

Table 6. Computing time required for the SST.

	N=3, l=4 m=4	N=7, l=5 m=5	N=3, l=10 m=10
Computing time	0.22s	T=109.94s	79.49s

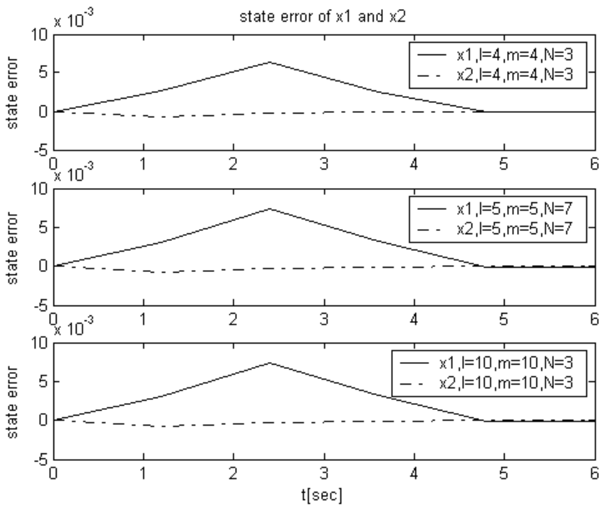

Fig. 4. State error response of the system for case 3.

the solution. But at the same time, there is a major increase in the amount of computing time required.

For our final case, we selected $T = 0.6$sec and $D = 0.25$sec . First we used a single-step Taylor method with $N = 2, 7, 10$ respectively. The results of states x_1 and x_2 for the Taylor method and Matlab solution are shown in Table 4 respectively.

In this case T and D are comparable to $2/\rho$ and therefore, as previously stated, it was difficult to obtain the desired accuracy using only a high order single-step Taylor method. From Table 4 the results of the Taylor series method were not useful even when the Taylor order N was increased to 10. In addition, 1223.65s were required to obtain the solution. In this case it is advisable to use the SST discretization scheme. From the definitions given in the previous section $q = 0$ and $\gamma = 0.25$. Assuming $l = m_\gamma$ and $m = m_{T-\gamma}$ are the scaling and squaring coefficients m of the two time intervals of $[kT, kT+\gamma)$ and $[kT+\gamma, kT+T)$. We then chose $N = 3,\ l = 6,\ m = 6$; $N = 7,\ l = 5,\ m = 5$ and $N = 3,\ l = 10, m = 10$ respectively. The results of states x_1 and x_2 of the Taylor method and Matlab solutions are shown in Table 5 respectively. Differences in response for these three methods are shown in Fig. 4. The computing time required for ten steps with these parameters is presented in Table 6. From these results, it is more efficient to use the SST when the sampling period T is large. Caution is required when selecting the SST parameters to save computing time.

5. CONCLUSIONS

This paper presented an approach to obtain discrete-time representations of nonlinear control systems with non-affine time-delay inputs in their control schemes. It was based on the ZOH assumption and the Taylor series expansion, which was obtained as a solution of continuous-time systems. The mathematical structure of the new discretization scheme was explored and characterized as useful for establishing concrete connections between numerical and system-theoretic properties. In particular, the effect of the time discretization method on key properties of nonlinear control systems with non-affine input time delays, such as equilibrium properties and asymptotic stability, was examined. The well-known scaling and squaring technique was expanded to nonlinear cases when the sampling period was too large. The proposed scheme provided a finite-dimensional representation for nonlinear systems with non-affine time-delay inputs enabling existing controller design techniques to be applied. The performance of the proposed discretization scheme was evaluated using a nonlinear system. Various sampling rates and time-delay values were considered, demonstrating the accuracy of the proposed discretization scheme.

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