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Switching Control for Second Order Nonlinear Systems Using Singular Hyperplanes

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Abstract: In this paper, we propose a switching control method for a class of 2nd order nonlinear systems with single input. The main idea is to switch the control law before the trajectory of the solution arrives at singular hyperplanes which are defined by the denominator of the control law. The proposed method can handle a class of nonlinear systems which is difficult to be stabilized by the existing methods such as feedback linearization, backstepping, control Lyapunov function, and sliding mode control.

Keywords: 2nd order nonlinear systems, switching control, singular hyperplanes, zero attraction regions, invariant set.

1. INTRODUCTION

Recently, there is growing attention to switching or hybrid control systems as an alternative to a single continuous feedback control [1,5,6,12,13,16,17]. The most serious obstacle to developing switching or hybrid schemes is unexpected behaviors caused by switching action such as chaotic transient response or even instability [8,9,10,15]. For example, it is well known that a family of subsystems may diverge in spite of the asymptotic stability of each subsystem [9,12,15].

Many efforts have been devoted to coping with such switching effects. One of the most representative methods is dwelling time analysis [1,6,9,10]. The main idea is that a switched system is stable if every respective subsystem is stable and switching is sufficiently slow to allow the transient effects to dissipate after each switching. However, this method is not applicable to a system with fast switching or under arbitrary switching since switching is not permitted for some time interval called dwelling time.

In this paper, we propose a switching control method which guarantees asymptotic stability regardless of unexpected behaviors caused by switching action. In the proposed method, the control

law is designed to guarantee the negative definiteness of the time derivative of Lyapunov function. The resulting control input is given as a rational function. So, it is not defined where the denominator of the rational function is equal to zero. To avoid this singularity, we switch the control input before the trajectory of the solution meets the singularity. The asymptotic stability of a given system is shown by the invariant property of a sector composed of the singular hyperplanes. The proposed method can handle a class of nonlinear systems which are difficult to be stabilized by the existing methods such as feedback linearization, backstepping, control Lyapunov function, and sliding mode control.

This paper is organized as follows. In Section 2, we present the motivation for this research and the issues to be discussed herein. In Section 3, a switching control scheme is proposed to overcome the problems mentioned in Section 2. We classify the applicable systems using a tangent cone in Section 4. Stability analysis is addressed in Section 5. In Section 6, the proposed scheme is compared with the existing methods such as feedback linearization, backstepping, control Lyapunov function, and sliding mode control. Finally, we conclude the paper with some remarks in Section 7.

2. PROBLEM STATEMENT

Consider 2nd order affine nonlinear systems with single input and a constant input matrix. Suppose that one of the dynamics of the system consists of only a drift term i.e. the control input is not involved in this scalar system as follows.

$$\dot{x} = f(x) + gu \quad \text{or} \quad \begin{cases} \dot{x}_1 = f_1(x) + g_1 u \\ \dot{x}_2 = f_2(x), \end{cases} \tag{1}$$

where $x = [x_1 \; x_2]'$, $g = [g_1 \; 0]'$, $f(x) = [f_1(x) \; f_2(x)]'$ is a smooth vector field with the property $f(0) = 0$, and g_1 is a constant. At least one of the equilibria of the system is the origin because $f(0) = 0$ and $u(0) = 0$, and suppose this is a distinct equilibrium point.

Consider a quadratic function $V = x'Mx$ for this system, where M is a symmetric positive definite matrix. Derivation of the function with respect to time along the trajectory of the solution of the system yields

$$\dot{V} = \dot{x}'Mx + x'M\dot{x}$$
$$= 2x'Mf(x) + 2x'M\,g\,u, \tag{2}$$

because M is a symmetric matrix and u is a scalar.

Define the control input as follows.

$$u = \frac{1}{x'Mg}\left[-x'Mf(x) - x'Mx\right]. \tag{3}$$

Since $\dot{V} = -2x'Mx < 0$ for any x except the origin by substituting (3) for (2), the origin of the system may be regarded as asymptotically stable unless the denominator of the rational function, $x'Mg$, is equal to zero. Here, we define a terminology called a singular hyperplane.

Definition 2.1 (Singular Hyperplane)**:** The set of points where the denominator of the control input (3) is equal to zero is the singular hyperplane generated by control input (simply singular hyperplane). That is, for given system (1) with control input (3)

$$S = \left\{ x \in \mathbb{R}^2 \mid x'Mg = 0,\; x \neq 0 \right\} \tag{4}$$

is the singular hyperplane with Mg as its normal vector.

To avoid a situation where the trajectory of the solution meets the singular hyperplane, the control input should be changed by selecting another M before the state trajectory arrives at the singular hyperplane. Unfortunately, the asymptotic stability of the origin cannot be guaranteed by only avoiding the singular hyperplane since there is the discontinuity of Lyapunov functions induced by replacing M and an unexpected behavior caused by switching action. Hence, it is necessary to develop a method which makes the trajectory avoid the singular hyperplane and stabilizes the system simultaneously in spite of unexpected behaviors caused by switching action as well as the discontinuity of Lyapunov functions.

3. SWITCHING CONTROL LAW

3.1. Control law

In this section, we propose a switching control scheme which stabilizes a class of 2nd order nonlinear

systems given in (1) by means of the control law given in (3). Consider again a system in (1) with a discontinuous control input as follows

$$\begin{cases} \dot{x}_1 = f_1(x) + u, \\ \dot{x}_2 = f_2(x), \end{cases}$$
$$u = \frac{1}{x'M_k g}\left[-x'M_k f(x) - x'M_k x\right],\; k = \{1,2\}, \tag{5}$$

where g is normalized into $[1\;0]'$ to simplify the problem. The control input u differs from (3) in that it is not continuous anymore because M_k in (5) is to be changed according to switching signal.

Note that u is well defined around the origin from the fact that the convergence rate of the numerator of the control input is faster than that of the denominator because the orders of the numerator and the denominator with respect to zero are 2 and 1 around the origin, respectively.

3.2. Switching rules

As mentioned in Section 2, u should be changed by switching M_k among appropriate candidates of positive definite matrices just before the trajectory of the solution arrives at the singular hyperplane defined in Definition 2.1. The problems are deciding when to change M_k and how to determine appropriate M_k.

With respect to the first problem, which is when it is the right time to change M_k, is related to the condition that generates switching signal. In this paper, we present two switching rules which trigger a switching action when at least one of them is satisfied.

Switching Rule 1: M_k is switched whenever the trajectory is sufficiently close to the singular hyperplane as follows.

$$\frac{|x'M_k g|}{\|x\|_2 \|M_k g\|_2} < \varepsilon \tag{6}$$

for some constant $\varepsilon > 0$, where $|x'M_k\,g|/\|M_k\,g\|_2$ means the distance between the current state x and the plane whose normal vector is $M_k\,g$. $\qquad\square$
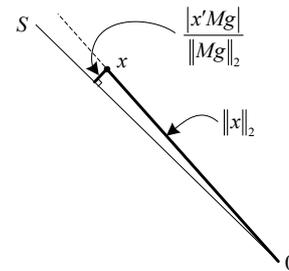

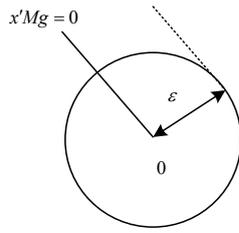Fig. 1. The switching rule by the ratio of distances.

Fig. 2. Distance trigger carries chattering around the origin.

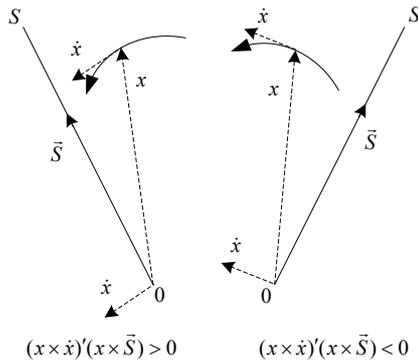$$(x \times \dot{x})'(x \times \vec{S}) > 0 \qquad (x \times \dot{x})'(x \times \vec{S}) < 0$$

Fig. 3. Motion direction of the trajectories.

This condition implies that if the ratio of the 6distance from the current state to the origin and to the singular hyperplane is less than ε, a switching action occurs as shown in Fig. 1.

The reason for choosing such distance ratio is chosen as a trigger of a switching action is to avoid chattering around the origin when simple distance trigger is employed as shown in Fig. 2.

We need the other switching condition as well as (6) to guarantee the invariant property of a sector as follows.

Switching Rule 2: M_k is switched whenever the trajectory does not head for the current activated singular hyperplane. Whether the trajectory heads for the current activated singular hyperplane or not is checked by

$$(x \times \dot{x})'(x \times \vec{S}_k) < 0, \qquad (7)$$

where $\vec{S}_k$ denotes the tangent vector of the corresponding singular hyperplane and × denotes the cross product of vectors.   □

As shown in Fig. 3, $(x \times \dot{x})'(x \times \vec{S}_k)$ is larger than 0 when the trajectory of the solution approaches to a singular hyperplane, while it is less than 0 when the trajectory recedes from a singular hyperplane. This condition indicates that the activated singular hyperplane always lies in front of the motion direction of the trajectory. By switching rules given in (6) and (7), we can show that the sector consisting of singular

hyperplanes is an invariant set in Section 5.

With respect to the second problem, which is how to determine appropriate M_k, is more complicated because the switching action may lead to unexpected behaviors in the system such as chaotic transient response and even instability [8-10,12]. For example, a switched system consisting of stable subsystems may become unstable under a certain switching rule [9,15].

In this paper, selecting M_k is directly related to the singular hyperplanes because $M_k g$ is the normal vector of the singular hyperplane. Thus, we need to define first some terminologies as follows: attraction region (AR), zero attraction region (ZAR), and sector.

Definition 3.1 (Attraction Region)**:** The region where the product of x_2 and the drift scalar system $f_2(x)$ in (5) is less than zero is called the attraction region for x_2. That is,

$$AR = \left\{ x \in \mathsf{R}^2 \mid x_2 f_2(x) < 0 \right\} \qquad (8)$$

is the attraction region of the system (5) with respect to x_2.

On the attraction region, the trajectory always heads toward the x_1 axis because x_2 and $\dot{x}_2 = f_2(x)$ have the opposite sign each other.

Definition 3.2 (Zero Attraction Region)**:** The attraction region in touch with the origin is called a zero attraction region and is abbreviated as ZAR. □

Definition 3.3 (Sector)**:** Each unilateral region between below a singular hyperplane and above the other one is called the sector generated by those singular hyperplanes and is referred simply as sector.□

Consider the situation where M_1 and M_2 are assigned for which the sector consisting of singular hyperplanes determined by M_1 and M_2 lies on ZAR. If the sector is an invariant set, the trajectory of the solution converges to the origin once it enters the sector.

Remark: In case of 2nd order systems, the existence of available ZAR in Step 1 can be easily checked by plotting AR on phase plane. We propose a criterion for the existence of available ZAR in Section 4 because it is hard to plot AR in case of higher order systems.

Table 1. The procedure of the proposed control scheme.

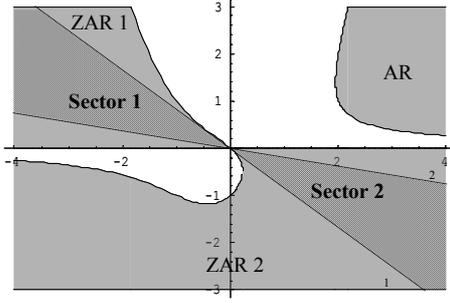| |
|---|
| Step 1. Check the existence of an available ZAR of a given system. |
| Step 2. Assign M_1 and M_2 for which the sector lies on ZAR at least around the origin. |
| Step 3. Switch the control laws in (5) according to the switching rules, (6) and (7). |

Fig. 4. Attraction region for $\dot{x}_2 = x_1 + x_2 + x_2^2 - x_1^2 x_2$.

Fig. 5. Tangent cone of region and boundary.

Remark: To assign sectors into ZAR at least around the origin in Step 2, it is sufficient to put the singular hyperplanes which is composed by M_k between the tangent planes of ZAR at the origin. More systematic procedure to assign M_k is discussed in Section 4.

For example, consider a system whose drift scalar system is given as follows.

$$\dot{x}_2 = x_1 + x_2 + x_2^2 - x_1^2 x_2$$

Plotting AR and ZAR on the phase plane for this system yields Fig. 4. This system has ZAR which are suitable for assigning sectors consisting of two singular hyperplanes, S_1 and S_2.

In this procedure, the control law is directly determined by the singular hyperplanes as shown in (5). Switching action between the control laws is employed to enclose the trajectory of the system in the sector. Then, each singular hyperplane plays the role of a barrier which does not permit the trajectory to cross them. The reason for this phenomenon will be discussed later by showing the sector is an invariant set. Finally, the terminal point of the trajectory admits only the origin because the trajectory never escape from the corn shaped sector as well as x_2 converges to zero on the sector included in ZAR.

4. APPLICABLE SYSTEMS

As mentioned in Table 1, an available ZAR must exist to apply the proposed switching control scheme. In this section, we propose a criterion of whether such ZAR exists or not. First, employ a terminology called tangent cone as follows.

Definition 4.1 (Tangent Cone)**:** Consider a subset $R \subset \mathsf{R}^n$ and a vector $x_0 \in R$. A vector $y \in \mathsf{R}^n$ is said to be a tangent vector of R at x_0 if either $y = 0$ or there exist a sequence $\{x_k\} \subset R$ and a non-negative sequence $\{a_k\}$ such that

$$\begin{cases} x_k \to x_0 \\ a_k(x_k - x_0) \to y. \end{cases}$$

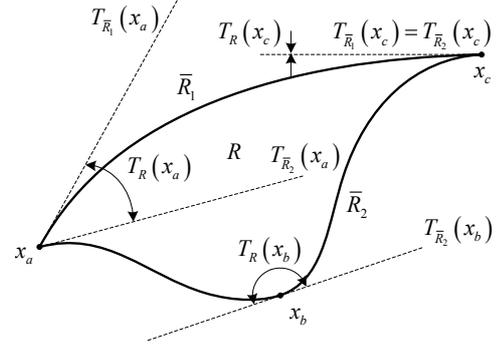The set of all tangent vectors of R at x_0 is called the tangent cone of R at x_0, and is denoted by $T_R(x_0)$. □

Fig. 5 shows the geometric description of the tangent cones of a region and its boundaries, where $\overline{R}_1$ and $\overline{R}_2$ are the smooth boundaries of R. At a point on a smooth boundary such as x_b, the boundary of the tangent cone of R at x_b coincides with the tangent cones of $\overline{R}_2$ at x_b. On the other hand, at a cusp point on the intersection of different boundaries such as x_a, the tangent cone of R at x_a has volume i.e. the tangent cones of $\overline{R}_1$ and $\overline{R}_2$ at x_a are linearly independent. However, at a cusp point such as x_c, the tangent cone of R at x_c has no volume i.e. the tangent cones of $\overline{R}_1$ and $\overline{R}_2$ at x_c are linearly dependent.

ZAR should have volume in the vicinity of the origin to assign sectors included in ZAR. To check the existence of an available ZAR, it is sufficient to evaluate whether or not the tangent cone of the corresponding ZAR at the origin has volume. That is, there exist an available ZAR when the tangent cones of $\{x \mid x_2 = 0\}$ and $\{x \mid \dot{x}_2 = 0\}$ at the origin are linearly independent because $\{x \mid x_2 = 0\}$ and $\{x \mid \dot{x}_2 = 0\}$ are the boundaries of ZAR by Definition 3.1 and 3.2 as well as the origin is a cusp point of ZAR. Note that the normal vector of all tangent cones of $\{x \mid x_2 = 0\}$ is $[0\ 1]'$ because $\{x \mid x_2 = 0\}$ is the x_1 axis itself.

For example, consider the following system

$$\begin{cases} \dot{x}_1 = -\sin x_2 + u \\ \dot{x}_2 = x_1. \end{cases} \tag{9}$$

The normal vectors of tangent cones of $\{x \mid x_2 = 0\}$ and $\{x \mid \dot{x}_2 = 0\}$ for the above system at the origin are

$$\begin{bmatrix} \partial x_2 / \partial x_1 \\ \partial x_2 / \partial x_2 \end{bmatrix}_{x=0} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} \partial \dot{x}_2 / \partial x_1 \\ \partial \dot{x}_2 / \partial x_2 \end{bmatrix}_{x=0} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$
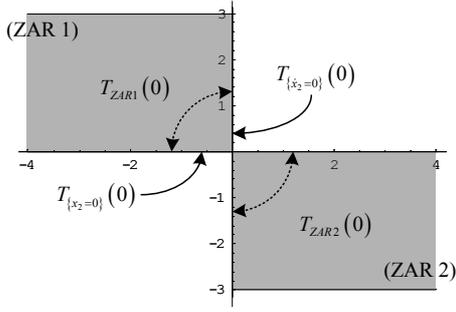
Fig. 6. Tangent cone and ZAR of $\dot{x}_2 = x_1$.

Fig. 7. Tangent cone and ZAR of $\dot{x}_2 = x_1 + x_1^3 x_2 + x_2^2$.
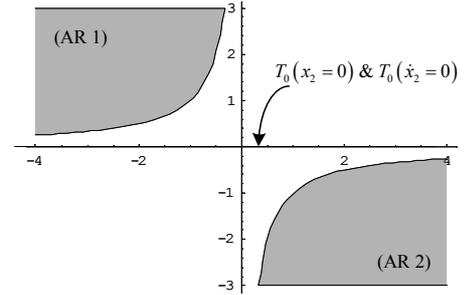
Fig. 8. Tangent cone and ZAR of $\dot{x}_2 = x_2 + x_1 x_2^2$.

In this case, there exist available ZAR because the normal vectors are linearly independent each other.

Fig. 6 depicts ZAR and the tangent cone of the system (9) at the origin. By Definition 3.1 and 3.2, ZAR is equivalent to the region where x_2 and $\dot{x}_2$ have the opposite sign each other, which corresponds to the tangent cone in the vicinity of the origin. Especially, in case that the dynamics of $\dot{x}_2$ is linear, the tangent cone coincides with zero attraction region as shown in Fig. 6.

In general, the tangent cone at the origin does not coincide with ZAR. For the following system

$$\begin{cases} \dot{x}_1 = x_1^2 x_2 + x_2 + u \\ \dot{x}_2 = x_1 + x_1^3 x_2 + x_2^2, \end{cases} \qquad (10)$$

the normal vectors of tangent cones of $\{x \mid x_2 = 0\}$ and $\{x \mid \dot{x}_2 = 0\}$ for the system at the origin are

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

which are equal to the case of the system (9). In this case, there exists available ZAR, while the tangent cone and ZAR does not coincide with each other because $\dot{x}_2$ dynamics is nonlinear as shown in Fig. 7.

The next example shows that the absence of ZAR implies that the tangent cone at the origin has no volume. This in turn means the fact that the tangent cone has volume is a sufficient condition for the existence of an available ZAR. For the system

$$\begin{cases} \dot{x}_1 = x_1 x_2 + x_1^2 + u \\ \dot{x}_2 = x_2 + x_1 x_2^2, \end{cases} \qquad (11)$$

the normal vectors of tangent cones of $\{x \mid x_2 = 0\}$ and $\{x \mid \dot{x}_2 = 0\}$ for the above system at the origin are linearly dependent as follows

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$
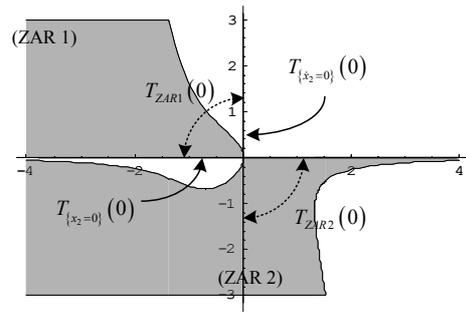
In this case, we can not assure the existence of ZAR Actually, ZAR for this system does not exist as shown in Fig. 8.

To determine that the proposed method is applicable to a given system, it is necessary to check whether or not the tangent cone of the system at the origin has volume. And to obtain appropriate M_k, it is sufficient to assign a sector in the tangent cone.

5. STABILITY ANANLYSIS

In this section, we discuss the stability issue of the proposed switching control system. As mentioned in the previous section, the state of the scalar system corresponding to drift dynamics, $f_2(x)$, converges to zero on ZAR. Hence, if the trajectory stays on ZAR, we can assert the stability of the scalar system i.e. the drift dynamics on which the control input is not directly imposed is asymptotically stable. Furthermore, if the trajectory stays on a sector included in ZAR, we can assert that both states, x_1 and x_2, converge to zero from the invariant property of the sector consisting of the singular hyperplanes. The following theorem shows that a sector consisting of singular hyperplanes is an invariant set.

Theorem 5.1: Assume that a switching action occurs whenever at least one of switching rules given in (6) and (7) is satisfied. Then, a sector consisting of the singular hyperplanes determined by M_k is an invariant set. (This is proven in Appendix A)   □

Note that Theorem 5.1 states that the sector is an

invariant set regardless of whether the sector lies on AR or not. The trajectory of the solution converges to the origin when it stays in the sector on ZAR. However, the trajectory diverges when it stays in the sector on non-AR.

Using this invariant property of sectors, we can show the asymptotic stability of the origin as the following theorem from the fact that the terminal point of the trajectory admits only the origin because once the trajectory enters a corn shaped sector, it can not escape from there and consequently x_2 converges to zero on the sector included in ZAR.

Theorem 5.2 If the tangent cones of $\{x \mid x_2 = 0\}$ and $\{x \mid \dot{x}_2 = 0\}$ at the origin for a given system are linearly independent and the switching rules in (12) are applied to a given system, then the origin of the proposed switching control system is asymptotically stable. (This is proven in Appendix C)   □

Note that this result addresses local stability. If both sectors are entirely included in ZAR, the origin of a given system is globally asymptotically stable.

6. COMPARISON WITH THE EXISTING METHODS

The proposed switching control method can handle a class of nonlinear systems which are difficult to stabilize with the existing methods. In this section, we compare the proposed scheme with the existing methods, in other to demonstrate the advantage of the proposed method.

6.1. Comparison with backstepping

Backstepping approach is not appropriate to the system (10) because the nonlinear terms of $\dot{x}_2$ dynamics is not a function of only x_2, i.e. it is not a triangular form.

On the other hand, the proposed switching control method is applicable to this system because the tangent cones, $T_{ZAR1}(0)$ and $T_{ZAR2}(0)$ of the system at the origin has volume or the tangent cones, $T_{\{x_2=0\}}(0)$ and $T_{\{\dot{x}_2=0\}}(0)$ are linearly independent each other as shown in Fig. 7. Therefore, the system (10) can be locally asymptotically stabilized by the proposed switching control method. The numerical result is given in Example 1 in Section 6.2.

6.2. Comparison with feedback linearization

Consider the system (10) rewritten as follows

$$\begin{cases} \dot{x}_1 = x_1^2 x_2 + x_2 + u \\ \dot{x}_2 = x_1 + x_1^3 x_2 + x_2^2. \end{cases}$$

To apply input-output feedback linearization to this

system, define x_2 as output and derive it as follows.

$$y = x_2 := \phi_1(x),$$

$$\dot{y} = \frac{\partial \phi_1}{\partial x}\dot{x} = \frac{\partial \phi_1}{\partial x}f(x) + \frac{\partial \phi_1}{\partial x}gu, \quad \frac{\partial \phi_1}{\partial x}g = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0.$$

Define $\phi_2(x) := \frac{\partial \phi_1}{\partial x}f(x) = x_1 + x_1^3 x_2 + x_2^2$,

$$\ddot{y} = \frac{\partial \phi_2}{\partial x}\dot{x} = \frac{\partial \phi_2}{\partial x}f(x) + \frac{\partial \phi_2}{\partial x}gu$$

$$= \left(1 + 3x_1^2 x_2\right)\left(x_1^2 x_2 + x_2\right) + \left(x_1^3 + 2x_2\right)\left(x_1 + x_1^3 x_2 + x_2^2\right)$$

$$+ \left(1 + 3x_1^2 x_2\right)u := v,$$

$$\ddot{y} = v.$$

The resulting control input u is not defined where $1 + 3x_1^2 x_2 = 0$. So, the trajectory can not cross this singularity as shown in Fig. 9.

On the other hand, the controllable region is extended by means of the proposed method as shown in Fig. 10. The numerical result is given in the following Example 1.

Example 1: Consider the system (10) whose attraction region is given in Fig. 7. We can assign appropriate singular hyperplanes because the tangent cones of the system at the origin are linearly independent from $[0\ 1]'$, $[1\ 0]'$ as their normal vector. In this case, the singular hyperplanes, S_1 and S_2 whose normal vector are [5 4]' and [1 2]' are assigned for the
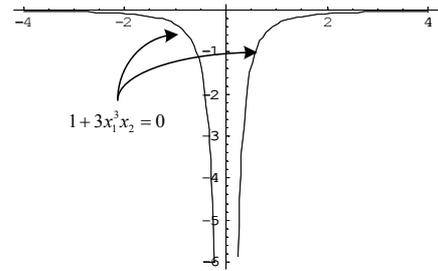

Fig. 9. Singularity caused by feedback linearization.
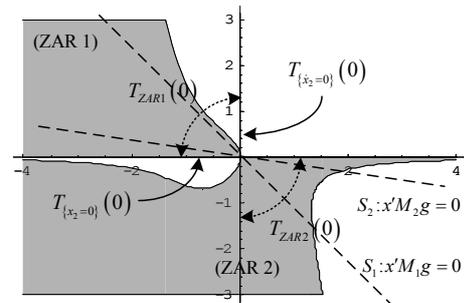

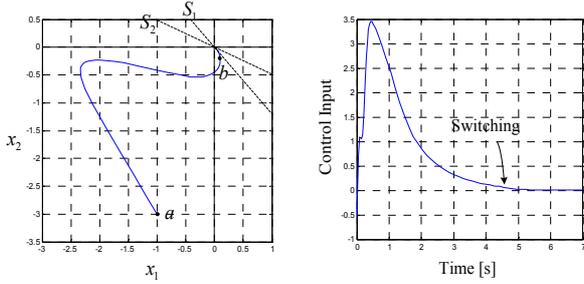Fig. 10. ZAR of the system (10).

Fig. 11. The trajectory and control input of Example 1 with initial point at (-1, -3).
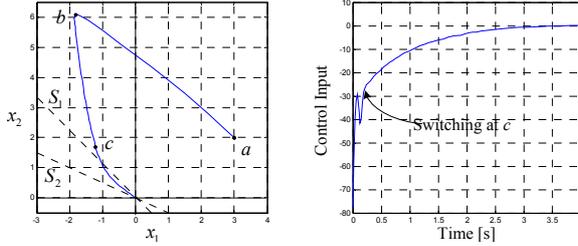

Fig. 12. The trajectory and control input of Example 1 with initial point at (3, 2).
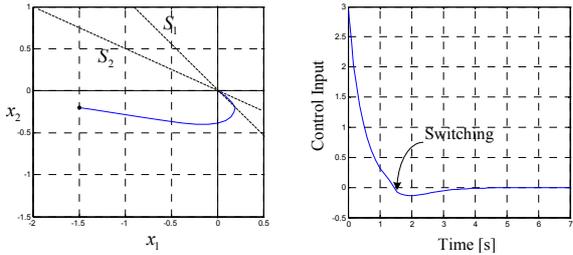

Fig. 13. The trajectory and control input of Example 1 with initial point at (-1.5, -0.2).

sectors to be included in ZAR as shown in Fig. 10. The corresponding positive definite matrices are assigned as $M_1 = [5\ 4;\ 4\ 4]$, $M_2 = [1\ 2;\ 2\ 5]$ because the first column of each matrix is the normal vectors of respective singular hyperplanes, and the distance ratio ε in (6) is set as 0.1.

Fig. 11 depicts the trajectory of the solution with the initial point at (-1, -2). The trajectory from (a) to (b) is dominated by S_1, and dominated by S_2 after (b). Finally, the trajectory enters the sector consisting of S_1 and S_2 and converges to the origin.

Compared with feedback linearization, this result shows that the proposed switching control extends the region of attraction. For example, when feedback linearization is applied to the system, the trajectory with the initial point at (-1, -2) can not converge to the origin because the trajectory of the solution can not cross the singularity as shown in Fig. 9. However, all trajectories with arbitrary initial points with the exception of the case where they start in non-AR on

the fourth quadrant converge to the origin by the proposed switching control method.

Note that non-AR laid across the first and second quadrants is the region of attraction. The reasons are that the trajectory with the initial point on such non-AR heads up during from (a) to (b) because $|x_2|$ increases in that region as shown in Fig. 12, and further the trajectory approaches S_1 like from (b) to (c) because the singular hyperplanes play the role of an attractor (See Appendix B). Finally, the trajectory enters the sector and converges to the origin. In addition, non-AR on the third quadrant is also the region of attraction by the similar analysis as shown in Fig. 13.

6.3. Comparison with control Lyapunov function
Consider the system (9) rewritten as follows

$$\begin{cases} \dot{x}_1 = -\sin x_2 + u \\ \dot{x}_2 = x_1. \end{cases}$$

Try an energy function $V(x) = x_1^2 + x_2^2$. The time derivative of the function is

$$\dot{V}(x) = 2x_1(-\sin x_2 + u) + 2x_1 x_2$$
$$= 2x_1(x_2 - \sin x_2 + u).$$

This function is not a Lyapunov function because $\dot{V} = 0$ when $x_1 = 0$. After some trial and error, try another candidate $V(x) = x_1^2 + 2x_2^2 + 2x_1 x_2$.

$$\dot{V}(x) = (2x_1 + 4x_2)x_1 + (2x_1 + 2x_2)(-\sin x_2 + u).$$

When $x_1 + x_2 \neq 0$, there exist u such that $\dot{V} < 0$. When $x_1 + x_2 = 0$, $\dot{V} = -2x_1^2 < 0$ unless $x_2 = 0$, which can only happen at $x_1 = -x_2 = 0$. Therefore the asymptotic stability can be achieved by the control input from this Lyapunov function.

This result thanks to the simple dynamics of $\dot{x}_2$. If $\dot{x}_2$ dynamics is given in more complicated form, it
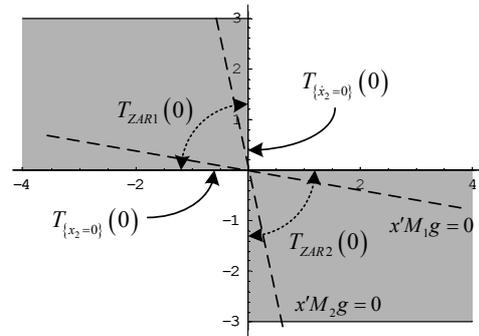

Fig. 14. Singular hyperplanes on the tangent cone.

would be not easy to find an appropriate Lyapunov function. On the other hand, with the proposed method, one need not go through trail and error to find a Lyapunov function and the asymptotic stability can be easily achieved by assigning singular hyperplanes on the tangent cone as shown in Fig. 14. The numerical result is given in Example 2 in Section 6.4.

6.4. Comparison with sliding mode control

Consider the following system

$$\begin{cases} \dot{x}_1 = -a_1 \sin x_2 - a_2 x_1 + u \\ \dot{x}_2 = x_1. \end{cases}$$

If the switching function is $s(t) = x_1(t) + m x_2(t)$, the reduced order sliding mode is

$$x_1(t) = -m x_2(t) \quad \text{or} \quad \dot{x}_2(t) = -m x_2(t),$$

where $m > 0$. The resulting control input which stabilize the system is given $u(t) = u_l(t) + u_n(t)$, where $u_l = a_2 x_1 - m x_1$ and $u_n = k x_2 - \eta \operatorname{sgn}(s(t))$.

This result also thanks to the simple dynamics of $\dot{x}_2$ as in the case of control Lyapunov function approach explained in Section 6.3. Furthermore, sliding mode control always carries chattering which is undesired phenomenon as shown in Fig. 15. In contrast, the proposed method can be applied easily by assigning singular hyperplanes on the tangent cone without worrying about chattering because $|x_2|$ always decreases in the sector as shown in Fig. 16.
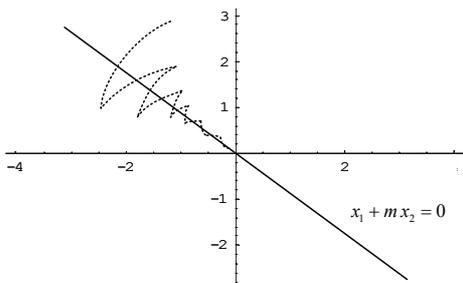

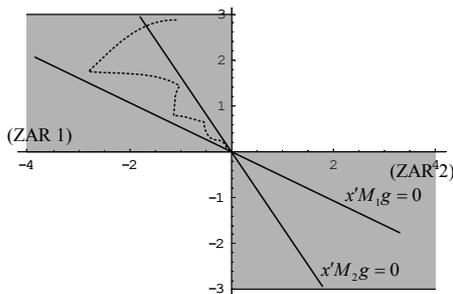
Fig. 15. Chattering in sliding mode control.

Fig. 16. Trajectory by the proposed control.

The numerical result of this system is similar to that of the system (9) in Example 2.

Example 2: Consider the system (9) whose attraction region is given in Fig. 6. We can assign appropriate singular hyperplanes because the tangent cones of the system at the origin are linearly independent with $[0\ 1]'$, $[1\ 0]'$ as their normal vector. In this case, the singular hyperplanes, S_1 and S_2 whose normal vector are [5 4]' and [1 2]' are assigned for the sectors to be included in ZAR as shown in Fig. 14. The corresponding positive definite matrices are assigned as $M_1 = [5\ 4;\ 4\ 4]$, $M_2 = [1\ 2;\ 2\ 5]$, and the distance ratio ε in (6) is set as 0.1.

Fig. 17 depicts the trajectory of the solution with initial point at (-3, -3). The trajectory from (a) to (b) is dominated by S_1, and dominated by S_2 after (b). Finally, the trajectory enters the sector consisting of S_1 and S_2 and converges to the origin.

The next example shows a case where control Lyapunov function and sliding mode control are not suitable for stabilizing the system.

Example 3: Consider the following system whose attraction region is given in Fig. 4.

$$\begin{cases} \dot{x}_1 = x_1^2 - x_1 x_2 + u \\ \dot{x}_2 = x_1 + x_2 + x_2^2 - x_1^2 x_2. \end{cases}$$

As shown in Fig. 4, we can assign singular hyperplanes for the sectors to be included in ZAR because the tangent cones of the system at the origin are linear independent with $[1\ 0]'$ and $[1\ 1]'$ as their normal vectors. In this case, the singular hyperplanes, S_1 and S_2 whose normal vector are [5 4]' and [1 2]' are assigned for the sectors to be included in ZAR. The corresponding positive definite matrices are assigned as $M_1 = [1\ 1;\ 1\ 3]$, $M_2 = [1\ 2;\ 2\ 5]$, and the distance ratio ε in (6) is set as 0.1.

Fig. 18 depicts the trajectory of the solution with initial point at (1, 1). By the proposed switching control, the sector is an invariant set where the trajectory can not escape from. Unlike sliding mode
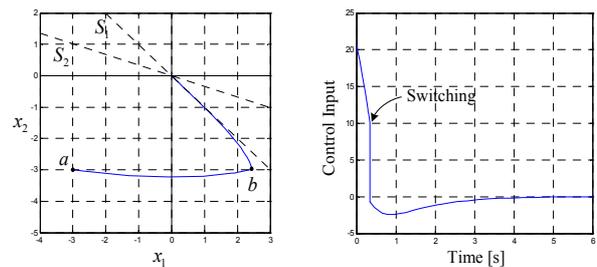

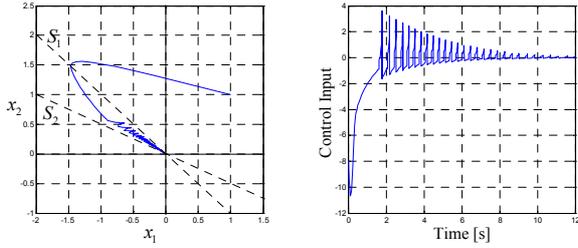Fig. 17. The trajectory and control input of Example 2.

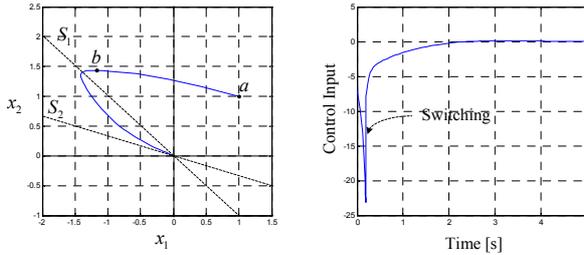Fig. 18. The trajectory and control input of Example 3.

Fig. 19. The trajectory and control input of Example 3.

control, the proposed switching control system does not carry chattering because $|x_2|$ always decreases in the sector on ZAR as time elapses. Furthermore, all trajectories of this system with arbitrary initial point converge to the origin because ZAR include both sectors entirely as shown in Fig. 4.

There is oscillation in the control input in Fig. 18, which is not a desired phenomenon. We can avoid such oscillation by increasing the interval between the singular hyperplanes. Fig. 19 shows the case where the normal vectors of the singular hyperplanes are assigned as $[1\ 1]'$ and $[1\ 3]'$, whose interval is wider than that of Fig. 18. The corresponding positive definite matrices are assigned as $M_1 = [1\ 1;\ 1\ 3]$, $M_2 = [1\ 3;\ 3\ 10]$, and the distance ratio ε in (6) is set as 0.1.

7. CONCLUSIONS

In this paper, we proposed a switching control method for a class of nonlinear systems which can not be handled with the existing methods such as feedback linearization, backstepping, control Lyapunov function approach, and sliding mode control.

The proposed method is similar to the inversion-based control such as feedback linearization in respect that some nonlinear dynamics are cancelled by control input. In addition, it is similar to the constructive Lyapunov function method such as backstepping in that the control law is directly induced from the derivative of Lyapunov functions. However, unlike feedback linearization, this method does not require the rank conditions and the involutiveness, and

contrary to backstepping, it is applicable to systems that do not have triangular forms. This method provides a simple way to design the control law comparing with control Lyapunov function approach and sliding mode control.

The proposed switching method introduces new concepts such as ZAR, sector, and tangent cone to show the asymptotic stability of the origin. Using tangent cone at the origin, the applicable systems can be easily checked. The sector on ZAR provides extended region of attraction because the trajectory in such sector never escape from there.

APPENDIX
A. Proof of Theorem 5.1

When the trajectory of the solution approaches a singular hyperplane,

$$(x \times \dot{x})'(x \times \vec{S}_k) > 0$$

because $x \times \dot{x}$ and $x \times \vec{S}_k$ have the same direction which heads up as shown on the left side of Fig. 3. However, when the trajectory recedes from a singular hyperplane,

$$(x \times \dot{x})'(x \times \vec{S}_k) < 0$$

because $x \times \dot{x}$ and $x \times \vec{S}_k$ have the opposite directions which head up and down, respectively as shown on the right side of Fig. 3. Hence, by the switching rule given in (7) the activated singular hyperplane always lies in front of the motion direction of the trajectory. That means the trajectory of (5) approaches to the current activated singular hyperplane S_2 as ② in Fig. 20.

Consider the situation just before switching action in the vicinity of S_2 because we can assign the distance ratio ε in (6) to be arbitrarily small. Then, the inner product of the normal vector of S_2 and the current vector field $\dot{x}_②$ is

$$[\alpha_2\ \ \gamma_2]\dot{x}_② < 0, \tag{12}$$

where $\dot{x}_②$ means the vector field when the current
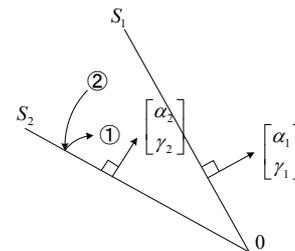

Fig. 20. The trajectory within sector.

singular hyperplane of the control input is S_2. Hence, the absolute value of the angle between $[\alpha_2 \ \gamma_2]'$ and $\dot{x}_②$ is larger than 90 degrees.

We will show that $[\alpha_2 \ \gamma_2]\dot{x}_① > 0$ when switching from S_2 to S_1 occurs in the vicinity of S_2 from the fact of $[\alpha_2 \ \gamma_2]\dot{x}_② < 0$, $[\alpha_2 \ \gamma_2]\dot{x}_① [\alpha_2 \ \gamma_2]\dot{x}_② < 0$ and that $[\alpha_1 \ \gamma_1]\dot{x}_② < 0$ when switching from S_1 to S_2 occurs in the vicinity of S_1 from the fact of $[\alpha_1 \ \gamma_1]\dot{x}_① > 0$, $[\alpha_1 \ \gamma_1]\dot{x}_② [\alpha_1 \ \gamma_1]\dot{x}_① < 0$. Hence, the absolute value of the angle between $[\alpha_2 \ \gamma_2]'$ and $\dot{x}_①$ is less than 90 degrees and that of between $[\alpha_1 \ \gamma_1]'$ and $\dot{x}_②$ is greater than 90 degrees imply that the trajectory heads for the interior of the sector as shown in Fig. 20 and Fig. 21, respectively.

Rewriting $[\alpha_2 \ \gamma_2]\dot{x}_②$ yields

$$[\alpha_2 \ \ \gamma_2]\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}_② = \alpha_2\dot{x}_1 + \gamma_2\dot{x}_2, \tag{13}$$

where the proposed control law in (5) is applied as follows.

$$\dot{x}_1 = f_1(x) + u$$
$$= \frac{1}{x'M_2g}\left[-x'M_2\begin{bmatrix} 0 \\ 1 \end{bmatrix}\dot{x}_2 - x'M_2x\right].$$

As it is in the above manipulation, the nonlinear term $f_1(x)$ is canceled, which is one of useful features of the proposed method. Replacing $\dot{x}_1$ in (13) by this result yields

$$[\alpha_2 \ \ \gamma_2]\dot{x}_② = \frac{\alpha_2}{x'M_2g}\left[-x'M_2\begin{bmatrix} 0 \\ 1 \end{bmatrix}\dot{x}_2 - x'M_2x\right] + \gamma_2\dot{x}_2$$
$$= \frac{x'M_2}{x'M_2g}\left\{\dot{x}_2\begin{bmatrix} \gamma_2 \\ -\alpha_2 \end{bmatrix} - \alpha_2x\right\}. \tag{14}$$

When switching occurs, the singular hyperplane S_2 is replaced by S_1. Hence, the vector field $\dot{x}_②$ is also changed into $\dot{x}_①$. The inner product of the normal vector of S_2 and $\dot{x}_①$ is

$$[\alpha_2 \ \ \gamma_2]\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}_① = \alpha_2\dot{x}_1 + \gamma_2\dot{x}_2,$$

where $\dot{x}_1 = [-x'M_1[0 \ 1]'\dot{x}_2 - x'M_1x]/x'M_1g$.

$$[\alpha_2 \ \ \gamma_2]\dot{x}_① = \frac{\alpha_2}{x'M_1g}\left[-x'M_1\begin{bmatrix} 0 \\ 1 \end{bmatrix}\dot{x}_2 - x'M_1x\right] + \gamma_2\dot{x}_2$$
$$= \frac{x'M_1}{x'M_1g}\left\{\dot{x}_2\begin{bmatrix} \gamma_2 \\ -\alpha_2 \end{bmatrix} - \alpha_2x\right\}. \tag{15}$$

Here, consider the product of (14) and (15) as follows.

$$[\alpha_2 \ \gamma_2]\dot{x}_② [\alpha_2 \ \gamma_2]\dot{x}_①$$

$$= \frac{1}{x'M_1g \cdot x'M_2g}\left\{\dot{x}_2\begin{bmatrix} \gamma_2 \\ -\alpha_2 \end{bmatrix} - \alpha_2x\right\}'$$

$$\left(M_2\begin{bmatrix} x_1^2 & x_1x_2 \\ x_1x_2 & x_2^2 \end{bmatrix}M_1\right)\left\{\dot{x}_2\begin{bmatrix} \gamma_2 \\ -\alpha_2 \end{bmatrix} - \alpha_2x\right\}.$$

Note $\dot{x}_2$ in (14) and (15) is equal to each other because the switching action affects only $\dot{x}_1$ but $\dot{x}_2$.

Since the switching action occurs in the vicinity of $S_2 = \{x \in \mathsf{R}^2 \mid \alpha_2x_1 + \gamma_2x_2 = 0\}$ and in an ideal case, it does not matter to consider that the switching action occurs on the singular hyperplane because the value of ε in (6) can be chosen to be arbitrarily small. So, replacing x_1 with $-(\gamma_2/\alpha_2)x_2$ yields (16)<0

$$[\alpha_2 \ \gamma_2]\dot{x}_② [\alpha_2 \ \gamma_2]\dot{x}_①$$

$$= \frac{1}{x'M_1g \cdot x'M_2g}x_2^2(x_2 + \dot{x}_2)^2\frac{\alpha_2\beta_2 - \gamma_2^2}{\alpha_2\beta_1}\left\{\left(\alpha_2 - \frac{\gamma_1}{\beta_1}\gamma_2\right)^2 + \frac{\gamma_2^2}{\beta_1^2}(\alpha_1\beta_1 - \gamma_1^2)\right\}$$
$$\tag{16}$$

because the current state lies inside the sector when switching occurs i.e. $x'M_1g \cdot x'M_2g < 0$ and $\alpha_2\beta_1 > 0$, $\alpha_1\beta_1 - \gamma_1^2 > 0$, $\alpha_2\beta_2 - \gamma_2^2 > 0$ because M_1, $M_2 > 0$. Hence, this result implies that

$$[\alpha_2 \ \ \gamma_2]\dot{x}_① > 0 \tag{17}$$

from the facts $[\alpha_2 \ \gamma_2]\dot{x}_② < 0$ in (12) and (16)<0. That means the absolute value of the angle between $[\alpha_2 \ \gamma_2]'$ and $\dot{x}_①$ is less than 90 degrees. Therefore, the trajectory heads for the interior of the sector by means of switching into S_1 in the vicinity of S_2.

In the opposite case of switching into S_2 in the vicinity of S_1, we can prove that

$$[\alpha_1 \ \ \gamma_1]\dot{x}_② < 0 \tag{18}$$

by the same manner. That means the absolute value of the angle between $[\alpha_1 \ \gamma_1]'$ and $\dot{x}_②$ is greater than 90 degrees as shown in Fig. 21.
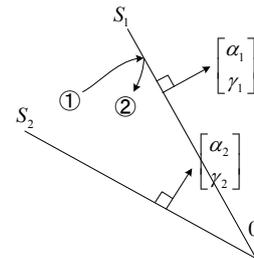

Fig. 21. The trajectory can not cross S_1.

These results of (17) and (18) imply that once the trajectory enters the sector consisting of S_1 and S_2, it can not escape from the sector by switching action. Therefore, the sector is a positively invariant set. □

B. Preliminary for Appendix C

Lemma: Each singular hyperplane plays a role of an attractor and has a temporary invariant set what is called the Lyapunov level surface.

Proof: Consider a fixed singular hyperplane. If there is no switching action in the future, the trajectory converges to the origin without touching the singular hyperplane because the time derivative of the Lyapunov function corresponding to the singular hyperplane is $\dot{V}_i = -x'M_k x < 0$. On the other hand, if there is a switching action in the future, at that time, the trajectory reaches the singular hyperplane with the exception of the origin. In both cases, the trajectory converges to the singular hyperplane. As long as a new singular hyperplane is not generated by a switching action, the trajectory can not escape from the Lyapunov level surface $V_i = x'(t_i)M_k x(t_i)$ because

$$\dot{V}_i = -x'M_k x < 0 \quad \text{for } t \in [t_i, \ t_{i+1}).\qquad \square$$

C. Proof of Theorem 5.2

Consider the Lyapunov level surface L_1 corresponding to the singular hyperplane S_1. By the first assumption of the theorem, there exist available ZAR which includes sectors at least around the origin. Hence, there is a ball B_1 that includes L_1 when the initial point of the trajectory is sufficiently close to the origin, where the radius of B_1 is given by the shortest distance from the origin to the intersection of sectors and the boundary of ZAR.

By Lemma in Appendix B, the current singular hyperplane S_1 acts as an attractor. So, the trajectory reaches S_1 in finite time and belongs to the sector consisting of S_1 and S_2. Note that the motion of the trajectory in the opposite direction is not permitted because x_2 increases on non-AR in Fig. 22. By switching rules (11) in Theorem 5.1, once the trajectory enters the sector, it cannot escape form the sector. Here, define a new ball B_2, where the radius of B_2 is given by the distance form the origin to the state when the trajectory reaches S_1. Then, B_2 belongs to B_1. By the same manner, define balls whenever the trajectory strikes S_1 as illustrate in Fig. 23.

Because each ball belongs to the previous one, these balls including the trajectory shrink to the origin as time elapses. Therefore, all trajectories starting within B_1 converge to the origin.

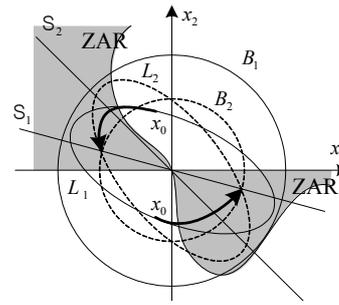Intuitively speaking, this theorem states that the

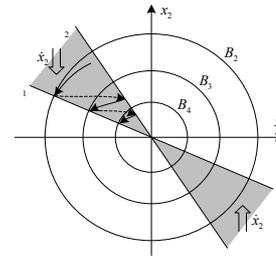Fig. 22. The trajectory belongs to B_1 and $B_2 \subset B_1$.

Fig. 23. Each ball belongs to the previous one.

terminal point of the trajectory admits only the origin because the trajectory cannot escape from the corn shaped sector as well as x_2 converges to zero on the sector included in ZAR. □

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