Error Reduction of Sliding Mode Control Using Sigmoid-Type Nonlinear Interpolation in the Boundary Layer

Yoo-Kyung Kim and Gi Joon Jeon

Abstract: Sliding mode control with nonlinear interpolation in the boundary layer is proposed. A modified sigmoid function is used for nonlinear interpolation in the boundary layer and its parameter is tuned by a fuzzy controller. The fuzzy controller that takes both the sliding variable and a measure of chattering as its inputs tunes the parameter of the modified sigmoid function. Owing to the decreased thickness of the boundary layer and the tuned parameter, the proposed method has superior tracking performance than the conventional linear interpolation method.

Keywords: Boundary layer, fuzzy controller, modified sigmoid function, sliding mode control.

1. INTRODUCTION

Sliding mode control is a robust nonlinear feedback control technique with the drawback of chattering. The most common method for solving the chattering problem is to introduce a boundary layer (BL) around the switching surface and to use continuous control inside the BL. This method, however, does not ensure the convergence of the state trajectory of the system to the sliding surface, and probably results in the existence of the steady-state error.

An alternative way to solve the chattering problem is to use the fuzzy sliding mode controller (FSMC), which combines a fuzzy controller (FC) with the sliding mode controller (SMC). The nonlinear transfer characteristic of the FSMC can be made to piecewise S-shape or reverse S-shape depending on the shape of the membership functions or its location [1,2]. The advantage of the FSMC is that the control method achieves asymptotic stability of the closed–loop system [3]. Also, the FSMC requires fewer fuzzy rules than the FC does. Due to the similarity between the FC and the SMC, the SMC has been used to guarantee the stability and robustness of the FC systems [4]. Although the FSMC is an effective method, its drawback is that the fuzzy rules must be previously tuned by time-consuming trial-and-error procedures. Furthermore, how to create suitable fuzzy rules remain uncertain. To overcome this problem, Lin and Hsu [5] proposed self-learning fuzzy sliding mode control, which can automatically adjust the fuzzy rules by a tuning algorithm. Lu and Chen [6] developed a self-organizing FSMC to achieve rapid and accurate tracking control of a class of nonlinear systems. Berstecher *et al.* [1] proposed an adaptive FSMC to better cope with changing system dynamics, unknown model uncertainties, and disturbances. However, the works [1,5-6] require an additional adaptive algorithm to improve control performance, thus it is likely to make the system complex and time-consuming.

In this paper, a modified sigmoid function (MSF), which is one of the reverse S-shaped functions, is proposed to interpolate nonlinearly in the BL and the parameter of the MSF is tuned by an FC as shown in Fig. 1. The fuzzy rules are employed to control the shape of the MSF by taking both the sliding variable, which is a measure of the distance to the sliding surface, and a measure of chattering as fuzzy input variables of the FC. This work has been motivated by the fact that the shape of the MSF can be made easily similar to the reverse S-shape transfer characteristic of the adaptive FSMC [1] and controlled simply by a single parameter. The sliding error of the proposed method is significantly reduced due to the narrowed virtual boundary layer (VBL) and the tuned shape of the MSF, which is the merit of the proposed method. Computer simulation examples using the proposed method for a DC motor and a nonlinear model are executed to show the performance of the proposed algorithm.

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2. SLIDING MODE CONTROL

Consider a single-input second order system

$$\ddot{x} = f(\boldsymbol{x}) + u + d, \tag{1}$$

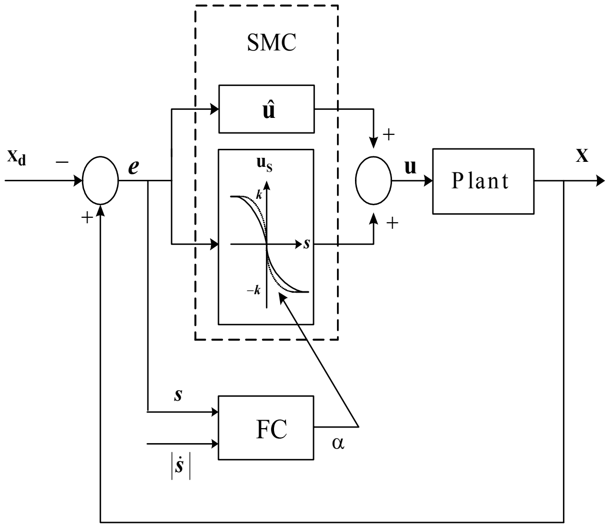


Fig. 1. Configuration of the control systems.

where the scalar x is the output of interest, $\boldsymbol{x}=[x\ \dot{x}]^T$ is the state vector, $f(\boldsymbol{x})$ is an unknown function, and the scalar u is the control input.

We assumed that the function $f(\boldsymbol{x})$ is

$$f(\boldsymbol{x})=\hat{f}(\boldsymbol{x})+\Delta f(\boldsymbol{x}), \tag{2}$$

where $\hat{f}(\boldsymbol{x})$ and $\Delta f(\boldsymbol{x})$ are the estimation of $f(\boldsymbol{x})$ and the model uncertainty, respectively. The model uncertainty and the disturbance are assumed to be bounded as

$$|\Delta f(\boldsymbol{x})|=|f(\boldsymbol{x})-\hat{f}(\boldsymbol{x})|\le F(\boldsymbol{x})\ \ and\ \ |d|\le D. \tag{3}$$

Let the time varying sliding surface s be expressed in the state-space $\mathbf{R}^2$ by $s(\boldsymbol{x};t)=0$ as

$$s(\boldsymbol{x};t)=(\frac{d}{dt}+\lambda)e=\dot{e}+\lambda e\ ,\quad \lambda>0 \tag{4}$$

and define the tracking error $\boldsymbol{e}=\boldsymbol{x}-\boldsymbol{x}_d=[e\ \dot{e}]^T$ where $\boldsymbol{x}_d=[x_d\ \dot{x}_d]^T$ is the desired state vector. The control input to allow the state x to track a specific time-varying desired state $\boldsymbol{x}_d$ in the presence of model uncertainty on $f(\boldsymbol{x})$ is made to satisfy the following sliding condition [7]:

$$\frac{1}{2}\frac{d}{dx}s^2\le-\eta\,|\,s\,|,\quad \eta\ge 0. \tag{5}$$

Let $\hat{u}$ be the nominal control law that can be interpreted as the best estimate, computed by $\dot{s}=0$ with known information $f=\hat{f}$, then it is found as follows:

$$\hat{u}=\ddot{x}_d-\lambda\dot{e}-\hat{f}. \tag{6}$$

Thus, the control law that satisfies the sliding mode condition in (5) can be obtained as

$$u=\hat{u}+u_s$$
$$u_s=-ksgn(s)\ ,$$
$$sgn(s)=\begin{cases}+1, & if\ s>0,\\ -1, & if\ s<0,\end{cases} \tag{7}$$

where $k\ge F+D+\eta$.

A certain disadvantage of this method is the drastic changes of the control input, which leads to high stress for the plant to be controlled. However, this can be avoided by means of the BL near the switching line, which smoothes out the control behavior and ensures that the system states remain within this layer

$$B(t)=\{\boldsymbol{x},\ |s(\boldsymbol{x};t)|\le\phi\},\qquad \phi>0\ , \tag{8}$$

where ϕ is the boundary layer thickness (BLT).

Therefore, we substitute the signum function $\operatorname{sgn}(s)$ in (7) by saturation function $sat(s/\phi)$.

$$u=\hat{u}-\bar{k}\ sat(s/\phi) \tag{9}$$

with $sat(x)=\begin{cases}x & if\ |x|<1\\ sgn(x) & otherwise,\end{cases}$ and $\bar{k}=k-\dot{\phi}$.

Also, the sliding condition in (5) is modified as follows;

$$\frac{1}{2}\frac{d}{dx}s^2\le(\dot{\phi}-\eta)\,|\,s\,|,\quad \eta\ge 0. \tag{10}$$

From (6), (7), and (10), the filter function follows [7]

$$\dot{s}=-\beta s+(-\Delta f(\boldsymbol{x}_d)+O(\varepsilon)),\ \ \beta\triangleq\frac{\bar{k}}{\phi}>0 \tag{11}$$

and $O(\varepsilon)$ represents a term of relatively small magnitude caused by using a desired state instead of the actual state vector in (11). This filter with bandwidth β removes the high-frequency chattering to give a smooth s.

Table 1. The system parameters of the motor.

Rule R[1]: If s is NB	then u is PB
Rule R[2]: If s is NM	then u is PM
Rule R[3]: If s is NS	then u is PS
Rule R[4]: If s is Z	then u is Z
Rule R[5]: If s is PS	then u is NS
Rule R[6]: If s is PM	then u is NM
Rule R[7]: If s is PB	then u is NB

3. FUZZY SLIDING MODE CONTROL

A FC that uses the sliding variable s as the input to calculate the control variable u belongs to the family of FSMCs. In [4], the FC is used to interpolate in BL. Due to the strictly monotonous decreasing u as s increases in the BL, the shape of the nonlinear transfer characteristic of the FSMC becomes S-shaped depending not only on the values of u, but also on the membership functions of the rule antecedents, consequents and defuzzification method. Similar to the SMC with BL, the rules are, in general, conditioned such that above the switching line a negative control output is generated with a positive one below it. The pattern of the control can be expressed by the numerous rules in Table 1, where NB denotes Negative Big, NM Negative Medium, NS Negative Small, Z Zero, PB Positive Big, PM Positive Medium, and PS Positive Small whose meanings are defined by corresponding membership functions.

Due to similarity between FCs and SMCs [2], we consider the following fuzzy sliding control law:

$$u = \hat{u} + u_f,$$
$$u_f = -k_{fuzzy}(e, \dot{e}, \lambda) \cdot sat(s/\phi), \tag{12}$$

where $k_{fuzzy}(e, \dot{e}, \lambda)$ is the absolute value of the control output of the FC. The nonlinear transfer characteristics of the above fuzzy rules, of which input and output membership functions are shown in Fig. 2, have a reverse S-shaped piecewise configuration as drawn in the same figure. The graph represents the control value u_f against s after the center-of-sums defuzzification.

4. NONLINEAR INTERPOLATION USING A SIGMOID FUNCTION

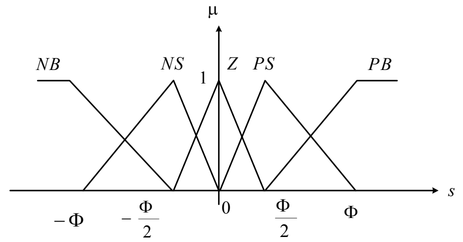
In order to eliminate the chattering caused by the signum function in (7) we introduce a MSF, which has the similar shape to the nonlinear transfer characteristics of the FSMC shown in Fig. 2(c), to replace the signum function when the states of the system get into the BL. The MSF used in this paper is

$$f(x) = -\frac{2}{1+e^{-\alpha x}} + 1, \tag{13}$$

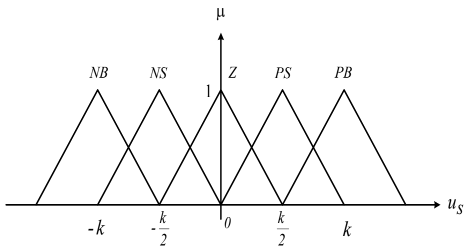
where α is a constant that controls the shape of the function. Thus, the new control law inside the BL is expressed as follows;

$$u_s' = \bar{k}\left(-\frac{2}{1+e^{-\alpha s}} + 1\right). \tag{14}$$

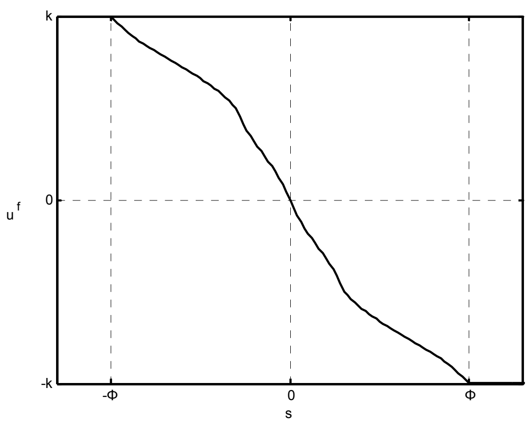
The nonlinear transfer characteristics of the SMC with the MSF inside the BL depend on the parameter α of the MSF. A FC is employed to tune the parameter α in (14) in order to control the shape of the MSF. The inputs of the FC are the sliding variable s and the measure of chattering Γ, which is defined as



(a) Input s.



(b) Output u_s.



(c) Nonlinear transfer characteristics of the FSMC.

Fig. 2. The Membership functions for the fuzzy sets.

Table 2. A set of the fuzzy rules to control the parameter of the MSF.

IF $\lvert s\rvert$ is zero and Γ is small	THEN $\Delta\alpha$ is PB
IF $\lvert s\rvert$ is zero and Γ is big	THEN $\Delta\alpha$ is PS
IF $\lvert s\rvert$ is small and Γ is small	THEN $\Delta\alpha$ is PB
IF $\lvert s\rvert$ is small and Γ is big	THEN $\Delta\alpha$ is PS
IF $\lvert s\rvert$ is big	THEN $\Delta\alpha$ is PS

$$\Gamma = |\dot{s}|. \tag{15}$$

From these inputs, the FC computes on-line the parameter of the MSF on the principles of the following fuzzy rules:

If the sliding variable is small and chattering is not occurring, increase parameter α *.*
If chattering is occurring, decrease parameter α *. regardless of the magnitude of the sliding variable.*

The control rules on this principle are expressed in Table 2, where we define $\alpha \triangleq \alpha_0 + \Delta\alpha.$ A parameter α is assumed to be in the following range:

$$\alpha_0 < \alpha < \alpha_\Gamma, \tag{16}$$

where α_0 is the starting value of the parameter α and α_Γ is the maximum value of the parameter before chattering occurs.

When the tuning of the parameter α is finished by the FC, the sliding control with MSF will be almost saturated in the wider range than the linear interpolation as shown in Fig. 3. Therefore, we can define the virtual boundary layer (VBL), $B'(t)$ as follows:

$$B'(t) = \left\{\boldsymbol{x},\ |s(\boldsymbol{x};t)| \le \phi'\right\}. \tag{17}$$

where $\phi' > 0$ is determined by elongating the MSF vertically as illustrated in Fig. 3. This adjustment is accomplished by multiplying the MSF by $\bar{k}'/\bar{k}'_c$, where

$$\bar{k}' = k - \dot{\phi}' \tag{18}$$

and $\bar{k}'_c$ is set to 99.9% of $\bar{k}'$ in this work.

The reason for this adjustment is that the candidate of the switching control u'_s in (14) converges to $-\bar{k}'$ in (18) only when s goes to infinity because of the characteristics of the MSF. Since the amount of adjustment is very small, the effect can be ignored. Now, the new control law becomes

$$u_s = \begin{cases} \bar{k}'\left(\dfrac{-2}{1+e^{-\alpha s}}+1\right)\dfrac{\bar{k}'}{\bar{k}'_c}, & |s| < \phi', \\ -\bar{k}'\,sgn(s), & \end{cases} \tag{19}$$

$$u = \hat{u} + u_s. \tag{20}$$

From (4) and (11), the structure of the error dynamics can be summarized as in Fig. 4 [1]. In this figure, the 2^{nd} order filter function can be rewritten as

$$e = \frac{1}{p+\lambda}\cdot\frac{\Delta f(\boldsymbol{x}_d) + O(\varepsilon)}{p+\beta'}. \tag{21}$$

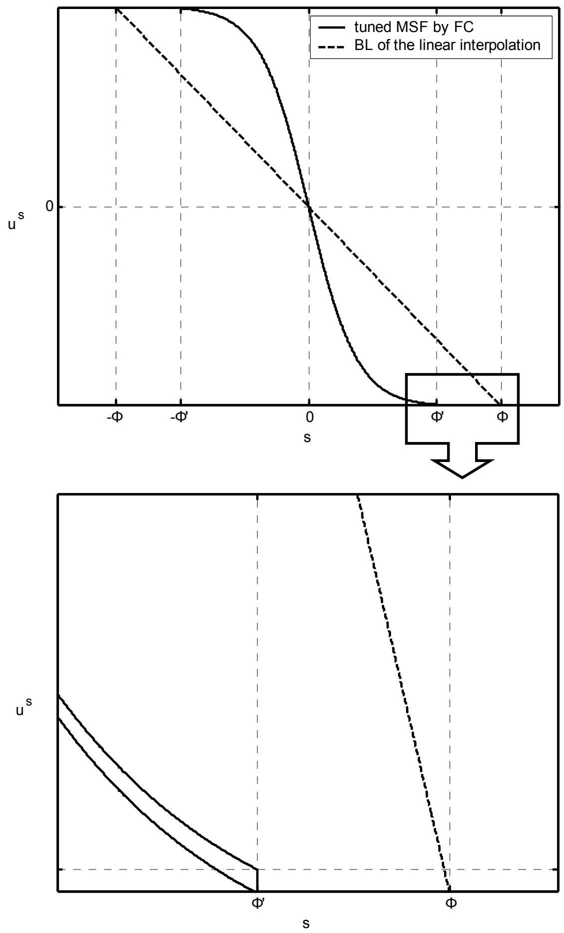


Fig. 3. BL of the linear interpolation and VBL after tuning.

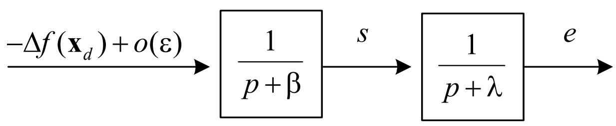


Fig. 4. Structure of the closed-loop dynamics.

When $k(x_d)$ is assumed to be dominant to $\dot{\phi}'$ or $\dot{\phi}$ it follows that

$$\beta' \triangleq \frac{k(x_d) - \dot{\phi}'}{\phi'} > \beta = \frac{k(x_d) - \dot{\phi}}{\phi}. \quad (22)$$

Since β' is greater than β the error dynamics (20) have a wider bandwidth than with β, which results in improved error reduction by forcing the pole in the more negative direction on the real axis. Thus, we can reduce the effect of uncertainty, $\Delta f(\boldsymbol{x}_d)$ without chattering.

Finally, we have the following design procedure.

- Procedure

Step 1: Choose k in (7).
Step 2: Calculate the parameter α_0 at the given BLT of linear interpolation.
Step 3: Obtain the parameter $\Delta\alpha$ from the FC.
Step 4: Adjust the switching control u_s in (19).
Step 5: Calculate the sliding mode control u from (20).

5. STABILITY

To design a stable SMC using MSF inside the BL, we must determine proper values of $\hat{u}$ and $\bar{k}'$ in (19). First, the approximation of control input, $\hat{u}$ would be obtained from (6). Next, we can choose the value of k by the bounds on f and d, and $\bar{k}'$ is obtained from (18).

Now, the following theorem ultimately gives the SMC system with MSF finiteness.

Theorem: For system (1), the proposed SMC with the MSF inside the BL makes the system trajectories ultimately bounded under the region given in (17), if we choose $\bar{k}'$ as

$$\bar{k}' \geq F + D + \eta - \dot{\phi}'. \quad (23)$$

Proof: Define the Lyapunov function

$$V = \frac{1}{2}s^2. \quad (24)$$

Then, when $|s| > \phi'$, the derivative $\dot{V}$ along the system trajectory is

$$\begin{aligned} \dot{V} &= s\,\dot{s} \\ &= s\,(f + d + u - \ddot{x}_d + \lambda\dot{e}) \quad (25) \\ &= s\,(f - \hat{f} + d + \bar{k}'\,sgn(s)) \end{aligned}$$

where (4), (6), and $u = \hat{u} - \bar{k}'\,sgn(s)$ are used. Thus,

$$\begin{aligned} \dot{V} &\leq s\,(F + D) - \bar{k}'|s| \\ &\leq -(\eta - \dot{\phi}')|s|, \end{aligned} \quad (26)$$

where (24) is utilized. The (26) satisfies the sliding condition, $\frac{1}{2}\frac{d}{dx}s(\boldsymbol{x},t)^2 \leq -(\eta - \dot{\phi}')\,|s|$ and it implies that $|s|$ is bounded by $|s| < \phi'$. Therefore, using the value $\bar{k}'$ of (21), we always have the bounded trajectories in the region of $B'(t)$ in (17).

6. ILLUSTRATIVE EXAMPLES

Example 1: A simplified model to simulate the dynamics of the DC motor has been developed using the classical equations [8]. A block diagram of the DC motor is shown in Fig. 5.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -(\frac{JR + K_f L}{JL})x_2 - (\frac{RK_f + K_m K_b}{JL})x_1 + \frac{K_m}{JL}v_a. \end{aligned} \quad (27)$$

where $x = [w(t), \dot{w}(t)]^T$, $w(t)$ is the angular velocity, v_a is the input voltage that drives the angular position output, J is the total moment of inertia reflected in the motor axis, T_d is the load disturbances and K_f is the viscous friction. The constants K_m, K_b, R and L given in Table 3 are the electrical constants of the motor. The sampling rate and the slope λ of the switching line are $0.01s$ and 5, respectively. Control law in (9) with BLT $\phi = 0.5$ is

$$u = \frac{(-14\dot{x} - 41x) + 5\dot{e}}{10} - 5\,sat(s/0.5). \quad (28)$$

Now, the MSF is used to interpolate nonlinearly within the BL. The control law in (19) is used as

$$u = \frac{(-14\dot{x} - 41x) + 5\dot{e}}{10} - 5\,(\frac{-2}{1 + e^{-\alpha s}} + 1)\,\frac{1}{0.999}. \quad (29)$$

To evaluate the nonlinear interpolation in the face of disturbance, we simulate the response to a step command $w_{ref} = 1$ with a disturbance $T_d = 0.1\ Nm$ between $t = 5$ and $t = 10$ seconds. The corresponding step response and control input are plotted in Figs. 6 and 7, respectively.

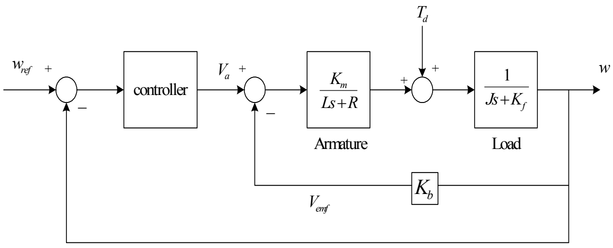


Fig. 5. DC motor model.

Table 3. The electrical constants of the motor used in the simulation experiment 1.

R	2.0 Ohm
L	0.5 Henry
K_m, K_b	0.1
K_f	0.2 Nm
J	0.02 $kg \cdot m^2 / s^2$

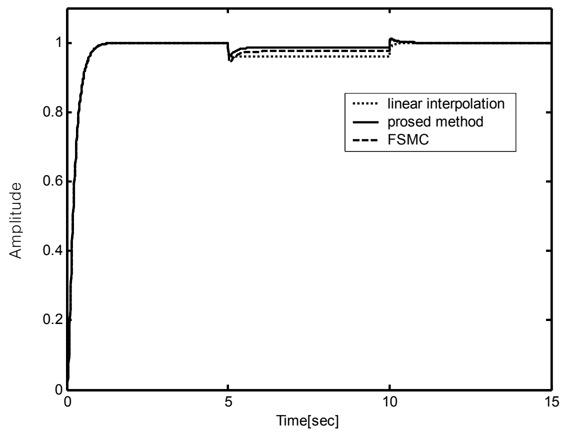


Fig. 6. Step response.

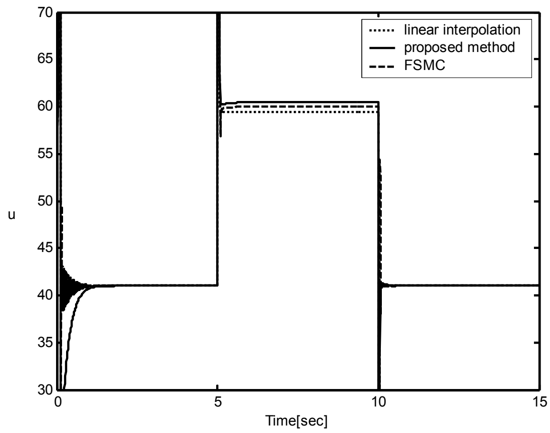


Fig. 7. Control input.

Example 2: A simplified nonlinear model of an underwater vehicle can be written as [7]

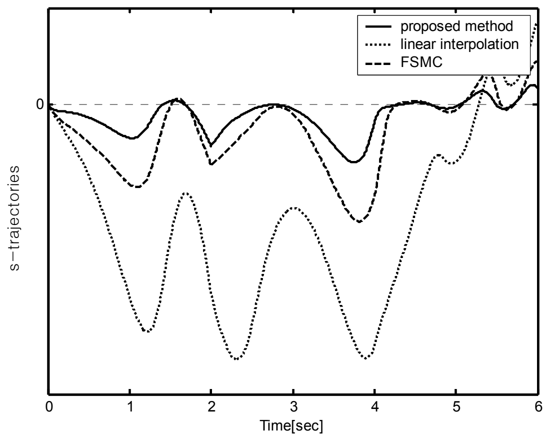


Fig. 8. s-trajectories with time-varying BLs.

$$m\ddot{x} + c\dot{x}|\dot{x}| = u,$$

$$\ddot{x} = -\frac{c\dot{x}|\dot{x}|}{m} + \frac{u}{m}, \quad 1 \le m \le 5, \quad 0.5 \le c \le 1.5,$$

$$f = -\frac{c\dot{x}|\dot{x}|}{m}, \quad b = \frac{1}{m}, \quad \beta = \sqrt{\frac{b_{max}}{b_{min}}} = \sqrt{5},$$

$$m = 3 + 1.5sin(|\dot{x}|t), \quad c = 1.2 + 0.2sin(|\dot{x}|t), \qquad (30)$$

$$\hat{f} = -\frac{\dot{x}|\dot{x}|}{\sqrt{5}}, \quad F = 0.3\dot{x}|\dot{x}|, \quad \hat{c} = 1,$$

$$\hat{m} = \sqrt{5}, \quad \eta = 0.1, \quad \lambda = 10,$$

$$x_d = sin(\pi t/2),$$

where x defines position, u is the control input, m is the mass of the vehicle, c is a drag coefficient, and the sampling rate is $0.002s$. The boundary ϕ is varying and the control law of (9) for this example is

$$u = c\dot{x}|\dot{x}| + \ddot{x}_d - 10\dot{e} - \bar{k}\, sat(s/\phi) \qquad (31)$$

with

$$k = 0.3\dot{x}|\dot{x}| + 0.1\sqrt{5} + \sqrt{5}(\sqrt{5}-1)\,|\,\ddot{x}_d - 10\dot{e}\,|,$$

$$k(x_d) \ge \frac{10\phi}{\sqrt{5}} \quad \Rightarrow \quad \dot{\phi} + 10\phi = \sqrt{5}\,k(x_d), \quad \bar{k} = k - \dot{\phi}/\beta,$$

$$k(x_d) \le \frac{10\phi}{\sqrt{5}} \quad \Rightarrow \quad \dot{\phi} + \frac{10\phi}{5} = k(x_d)/\sqrt{5}, \;\; \bar{k} = k - \dot{\phi}\,\beta.$$

When the MSF is used to interpolate in a varying boundary layer, the control law of (19) is

$$u = c\dot{x}|\dot{x}| + \ddot{x}_d - 10\dot{e} - \bar{k}'\left(\frac{-2}{1+e^{-\alpha s}} + 1\right)\frac{\bar{k}'}{\bar{k}_c'}, \qquad (32)$$

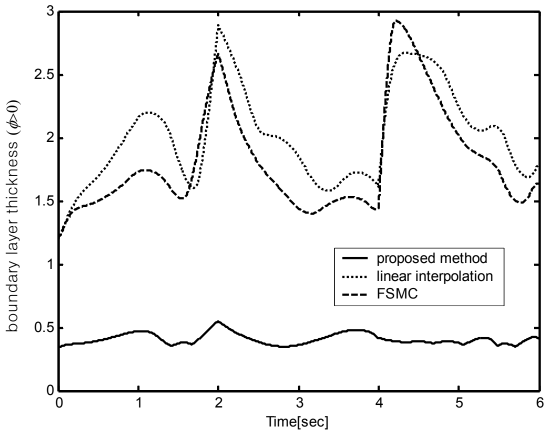


Fig. 9. Boundary layer thickness ($\phi > 0$).

where α is the output of the FC.

The proposed method is compared with two different types of interpolations, the conventional FSMC in section 3 and the SMC with linear interpolation in the BL. In Figs. 8 and 9, the s-trajectories and BLT of the above two cases and the proposed methods are shown, respectively. One can see that the s-trajectories of the proposed method are significantly reduced because of the narrowed BL.

6. CONCLUSIONS

The MSF has been employed for nonlinear interpolation within the BL as opposed to the conventional sliding mode controller with linear interpolation in the fixed BL or in the variable BL. After the operating range of the parameter of the MSF is updated on-line, the parameter is tuned by the FC. Due to the wide bandwidth of the error filter function, we can reduce the steady state error by the proposed SMC with the MSF under the system uncertainties and disturbances. By computer simulations, the proposed controller has shown to produce less significant steady-state errors than the conventional ones.

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