Necessary and Sufficient Stability Condition of Discrete State Delay Systems

Young Soo Suh, Young Shick Ro, Hee Jun Kang, and Hong Hee Lee

Abstract: A new method to solve a Lyapunov equation for a discrete delay system is proposed. Using this method, a Lyapunov equation can be solved from a simple linear equation and N-th power of a constant matrix, where N is the state delay. Combining a Lyapunov equation and frequency domain stability, a new stability condition is proposed for a discrete state delay system whose state delay is not exactly known but only known to lie in a certain interval.

Keywords: Time delay systems, stability, Lyapunov function, discrete systems.

1. INTRODUCTION

State delays are frequently encountered in control problems of many physical systems. In particular, continuous state delay systems have been received a lot of attentions and many stability results have been proposed (see [1-3] and its references). On the other hand, there has been less attention to the following discrete state delay system

$$x(k+1) = A_0 x(k) + A_1 x(k-N), \qquad (1)$$

where $x \in R^n$ is a state. The reason of less attention is not surprising since system (1) can be transformed into an equivalent non-delayed system. Introducing an augmented new state $z(k)$ as follows:

$$z(k) \equiv \begin{bmatrix} x(k) \\ \hline x(k-1) \\ \vdots \\ x(k-N) \end{bmatrix} \in \begin{bmatrix} R^{n\times 1} \\ R^{nN\times 1} \end{bmatrix},$$

we can obtain an equivalent non-delayed system

$$z(k+1) = A_N z(k), \qquad (2)$$

where

$$A_N \equiv \begin{bmatrix} A_0 & & A_1 \\ I & & \\ & \ddots & \\ & & I \end{bmatrix}.$$

Now stability of (1) can be investigated by simply checking stability of an ordinary non-delayed system (2).

However, there are two important cases that stability check of the equivalent non-delayed system is not adequate for stability check of (1). The first case is when the state delay N is very large, and the second case is the state delay N is not exactly known but only known to lie in a certain interval. If N is very large, then the A_N matrix of (2) is very large (note $A_N \in R^{n(N+1)\times n(N+1)}$). Hence stability check of A_N is numerically demanding and sometimes an unstable task. If the state delay N is only known to lie in a certain interval (for example, $N \in [0, N_{\max}]$), then stability should be checked for each $N \in [0, N_{\max}]$, which is also a numerically demanding task, in particular for large $N_{\max}$.

To cope with these two cases, we propose a new non-conservative stability condition of (1) by carefully investigating a Lyapunov equation for (2). In Section 2, it is shown that a solution of a Lyapunov equation for (2) can be transformed into a simple linear equation, where the only term depending on N is N-th power of a constant matrix. Hence, even for large N, the computation is simple. In Section 3, combining the constant matrix with frequency domain interpretations, we propose a stability condition for (1), where the state delay N is only known to lie in a certain interval. In Section 4, a numerical example is given to illustrate the results of this paper. In Section 5, conclusion is provided.

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The works which are most related to ours are [4-9]. In [4-7], delay independent stability conditions are considered. The conditions are, however, conservative when the state delay is known to lie in a certain interval. In [8], a robust stability problem is considered for a exactly known delay case. In [9], a delay dependent stability condition is derived based on Razumikhin-type theorem. In [10,11], similar delay dependent stability condition is used to derive a controller. All these delay dependent stability conditions are sufficient conditions, while in this paper the necessary and sufficient condition is proposed.

Notation is standard. For a matrix $M \in C^{n\times n}$ given by

$$M = \begin{bmatrix} m_{11} & \cdots & m_{1n} \\ \vdots & \vdots & \vdots \\ m_{n1} & \cdots & m_{nn} \end{bmatrix},$$

csM is defined by

$$csM \equiv [m_{11} \cdots m_{n1} \,|\, \cdots \,|\, m_{1n} \cdots m_{nn}]' \in C^{n^2 \times 1}.$$

Symbol $\otimes$ denotes the kronecker product.

2. LYAPUNOV FUNCTION

The Lyapunov function for (2) is defined by

$$V(z(k)) \equiv z(k)' P z(k),$$

where symmetric matrix $P \in R^{n(N+1)\times n(N+1)}$ is partitioned compatible to the partition of $z(k)$ and labeled as follows:

$$P = \begin{bmatrix} P_{00} & P_{01}(N) & \cdots & P_{01}(1) \\ P_{10}(N) & P_{11}(N,N) & \cdots & P_{11}(N,1) \\ \vdots & \vdots & \ddots & \vdots \\ P_{10}(N) & P_{11}(1,N) & \cdots & P_{11}(1,1) \end{bmatrix}.$$

It is standard from the following $V(z(k+1)) - V(z(k)) = z(k)'(A_N' P A_N - P) z(k)$ that the system (2) is stable if and only if there exists $P = P' > 0$ satisfying $A_N' P A_N - P < 0$. From the special structure of $z(k)$ ($z(k)$ is stable if and only if $x(k)$ is stable), the condition $A_N' P A_N - P < 0$ can be modified as in the following lemma.

Lemma 1: System (1) is stable if and only if there exists $P = P' > 0$ with $P_{00} > 0$ satisfying

$$A_N' P A_N - P + \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix} = 0 \tag{3}$$

for some $Q = Q' \in R^{n\times n} > 0$.

Proof: Note that (3) is not a standard Lyapunov form. Standard Lyapunov theorem [12] states that (1) is stable if and only if there exists $\tilde{P} = \tilde{P}' > 0$ satisfying

$$A_N' \tilde{P} A_N - \tilde{P} + \tilde{Q} = 0 \tag{4}$$

for some $\tilde{Q} = \tilde{Q}' \in R^{n(N+1)\times n(N+1)} > 0$.

We show that $P = P' \geq 0$ with $P_{00} > 0$ satisfying (3) exists if and only if $\tilde{P} = \tilde{P}' > 0$ satisfying (4) exists. To do this, we construct $\tilde{P}$ from P. Let $\hat{P} \in R^{n(N+1)\times n(N+1)}$ be defined by

$$\hat{P}(\varepsilon) \equiv \begin{bmatrix} 0 & & & & \\ & N\varepsilon I & & & \\ & & (N-1)\varepsilon I & & \\ & & & \ddots & \\ & & & & \varepsilon I \end{bmatrix},$$

$\varepsilon > 0$

where all unspecified entries are 0. Then,

$$A_N' \hat{P}(\varepsilon) A_N - P(\varepsilon) = \begin{bmatrix} N\varepsilon I & & & & \\ & -\varepsilon I & & & \\ & & -\varepsilon I & & \\ & & & \ddots & \\ & & & & -\varepsilon I \end{bmatrix}.$$

Let $\tilde{P} \in R^{n(N+1)\times n(N+1)}$ be defined by

$$\tilde{P}(\varepsilon) = P + \hat{P}(\varepsilon), \tag{5}$$

then,

$$A_N' \tilde{P}(\varepsilon) A_N - \tilde{P}(\varepsilon) = \begin{bmatrix} -Q + N\varepsilon I & & & & \\ & -\varepsilon I & & & \\ & & -\varepsilon I & & \\ & & & \ddots & \\ & & & & -\varepsilon I \end{bmatrix}. \tag{6}$$

((4) $\Leftarrow$ (3) part) If $P = P' \geq 0$ with $P_{00} > 0$ satisfying (3) exists, then $\tilde{P}$ is positive definite and satisfies (6). If we choose $\varepsilon > 0$ such that $-Q + N\varepsilon I < 0$, then (4) is satisfied.

((4) $\Rightarrow$ (3) part) Suppose $\tilde{P} = \tilde{P}' > 0$ satisfying (6) with $-Q + N\varepsilon I < 0$ and $\varepsilon > 0$ exists. Then by partitioning $\tilde{P}$ as in (5), we can show that P satisfies (3). Furthemore, since $\tilde{P}>0$, P_{00} ((1,1) element of $\tilde{P}$) is positive definite. Remaining to show is $P \geq 0$. Since $\tilde{P}>0$, for any $\tilde{x} \neq 0 \in R^{n(N+1)\times 1}$, the following is satisfied :

$$\begin{aligned}\tilde{x}'\tilde{P}(\varepsilon)\tilde{x} &= \tilde{x}'P\tilde{x} + \tilde{x}'\hat{P}(\varepsilon)\tilde{x} \\ &= \tilde{x}'P\tilde{x} + f(\tilde{x})\varepsilon > 0,\end{aligned} \tag{7}$$

where $f(\tilde{x}) \in R \geq 0$. Since (7) is satisfied for all ε satisfying $0 < N\varepsilon I \leq Q$, ε independent $\tilde{x}'P\tilde{x}$ term should be greater or equal to 0. Thus $P \geq 0$.

Matrix P has 3 variables: P_{00}, $P_{01}(i) = P_{10}(i)'$ and $P_{11}(i,j) = P_{11}(j,i)'$. The following lemma simplifies the expression of P using one variable $X(i)$.

Lemma 2: The solution P satisfying (3) is given by

$$\begin{aligned}&P_{00} = X(0),\\ &P_{01}(i) = X(i)A_1,\\ &P_{11}(i,j) = \begin{cases} A_1'X(i-j)'A_1, 0 \leq j \leq i \leq N,\\ A_1'X(j-i)A_1, 0 \leq i \leq j \leq N,\end{cases}\end{aligned} \tag{8}$$

where $X(k), 0 \leq k \leq N$ is given by

$$\begin{aligned}&A_0'X(0)A_0 + A_1X(N)'A_0 + A_0'X(N)A_1\\ &+A_1'X(0)A_1 - X(0) + Q = 0,\\ &X(0) = X(0)',\\ &X(k+1) = A_0'X(k) + A_1X(N-k)',\\ &0 \leq k \leq N-1.\end{aligned} \tag{9}$$

Proof: Straightforward calculation of (3) yields (8) and (9). □

Since the matrix difference equation (third equation) in (9) is not in an easy form to solve, the matrix difference equation is transformed into a kind of two point boundary value problem in the next lemma. Throughout the paper, A_0 is assumed to be nonsingular: we note that most discrete systems have nonsingular A_0 matrices.

Lemma 3: The matrix difference equation in (9) is equivalent to the following:

$$\begin{bmatrix} csX(k+1) \\ csX(N-k-1)\end{bmatrix} = \begin{bmatrix} A & B \\ -A^{-1}BA & A^{-1}(I-BB)\end{bmatrix}\begin{bmatrix} csX(k) \\ csX(N-k)\end{bmatrix}, \tag{10}$$

where $A \equiv (I \otimes A_0')$, $B \equiv (I \otimes A_1')T$, and

$$T \equiv [T_1 \mid T_2 \mid \cdots \mid T_{n^2}], T_l \in R^{n^2\times 1}. \tag{11}$$

Row vector $T_l, 1 \leq l \leq n^2$ is defined by

$$T_{(i-1)n+j} \equiv e_{(j-1)n+i}, 1 \leq i,j \leq n,$$

where $e_l \in R^{n^2\times 1}, 1 \leq l \leq n^2$ is a row vector whose l-th element is 1 and all other elements are 0.

Proof: Note the following properties of column string

$$\begin{aligned}&cs(ABC) = (C'\otimes A)csB,\\ &csA' = TcsA,\end{aligned} \tag{12}$$

where $A,B,C \in R^{n\times n}$. Using these properties, we obtain from the forward difference equation in (9):

$$csX(k+1) = AcsX(k) + BcsX(N-k). \tag{13}$$

Thus from the above equation, we obtain the backward equation (note that A is nonsingular if and only A_0' is nonsingular)

$$csX(k) = A^{-1}csX(k+1) - A^{-1}BcsX(N-k).$$

Letting $k \equiv N-k-1$ in the above equation, we obtain

$$\begin{aligned}csX(N-k-1) = -A^{-1}BAcsX(k) + \\ A^{-1}(I-BB)csX(N-k).\end{aligned} \tag{14}$$

Combining (13) and (14), we obtain (10). □

For later reference, we define two matrices H (see (10)) and J:

$$H \equiv \begin{bmatrix} A & B \\ -A^{-1}BA & A^{-1}(I-BB)\end{bmatrix}, J \equiv \begin{bmatrix} 0 & I \\ I & 0\end{bmatrix}. \tag{15}$$

Now return to the problem of solving (9) for some Q. The matrix Lyapunov equation (first equation) and the matrix difference equation (third equation) are coupled in (9). To solve the Lyapunov equation, it is necessary to obtain an $X(0)$ and $X(N)$ pair satisfying the matrix difference equation, or equivalently (10). The constraint imposed on any $X(0)$ and $X(N)$ pair satisfying (10) can be stated using the boundary condition:

$$\begin{bmatrix} csX(N) \\ csX(0)\end{bmatrix} = H^N \begin{bmatrix} csX(0) \\ csX(N)\end{bmatrix},$$

and the above equation can be expressed using J (see (15)) as follows:

$$(I - JH^N)\begin{bmatrix} csX(0) \\ csX(N) \end{bmatrix} = 0 . \tag{16}$$

Does an $X(0)$ and $X(N)$ pair satisfying (16) always exist? For example, if $\dim \mathrm{Null}(I - JH^N) = 0$, then only $[(csX(0))'(csX(N))']' = 0$ can be the solution to (16). The next lemma shows that $\dim \mathrm{Null}(I - JH^N) = n^2$ and thus there exists a nontrivial $X(0)$ and $X(N)$ pair satisfying (16).

Lemma 4: The following is satisfied.

$$\dim \mathrm{Null}(I - JH^N) = n^2 . \tag{17}$$

The proof of Lemma 4 needs the following lemma.

Lemma 5: If z is an eigenvalue of H, then z^{-1} is also an eigenvalue of H.

Proof: If z is an eigenvalue of H, there exists a $[x'y']' \neq 0$ satisfying

$$H\begin{bmatrix} x \\ y \end{bmatrix} = z\begin{bmatrix} x \\ y \end{bmatrix}.$$

The upper block of the above equation is

$$Ax + By = zx. \tag{18}$$

The lower block is

$$-A^{-1}BAx + A^{-1}(I - BB)y = zy. \tag{19}$$

Inserting (18) into (19), we obtain

$$z^{-1}y = Bx + Ay . \tag{20}$$

Now consider

$$H\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} Bx + Ay \\ -A^{-1}BAy + A^{-1}(I - BB)y \end{bmatrix}.$$

From (20), the upper block of the above equation is $z^{-1}y$. Using (18) and (20), the lower block is given by

$$-A^{-1}BAy + A^{-1}(I - BB)y = z^{-1}x .$$

Hence we obtain

$$H\begin{bmatrix} x \\ y \end{bmatrix} = z^{-1}\begin{bmatrix} x \\ y \end{bmatrix}. \tag{21}$$

From the proof of Lemma 5, we note that eigenvalues and eigenvectors of H (for simplicity, all eigenvalues of H are assumed to be simple and nonzero) can be expressed as

$$H\begin{bmatrix} X & Y \\ Y & X \end{bmatrix} = \begin{bmatrix} X & Y \\ Y & X \end{bmatrix}\begin{bmatrix} \Sigma & \\ & \Sigma^{-1} \end{bmatrix}, \tag{22}$$

where $\Sigma \in C^{n^2 \times n^2}$ is a diagonal matrix whose diagonal elements are eigenvalues of H.

Proof of Lemma 4: First we will show that

$$\dim \mathrm{Null}(I - JH^N) \geq n^2 . \tag{23}$$

Let v is defined by

$$v \equiv \begin{bmatrix} X & Y \\ Y & X \end{bmatrix}\begin{bmatrix} I \\ \Sigma^N \end{bmatrix}\alpha ,$$

where $\alpha \in C^{n^2 \times 1}$, then $v \in \mathrm{Null}(I - JH^N)$ from (22). Hence we have

$$\dim \mathrm{Null}(I - JH^N) \geq \dim \mathrm{Range}(\begin{bmatrix} X & Y \\ Y & X \end{bmatrix}\begin{bmatrix} I \\ \Sigma^N \end{bmatrix}) = n^2.$$

Similarly, we can show that

$$\dim \mathrm{Null}(-I - JH^N) \geq n^2 . \tag{24}$$

From (23) and (24), we can conclude that JH^N has only two eigenvalues ± 1 and thus

$$\dim \mathrm{Null}(I - JH^N) = 2n^2 - \dim \mathrm{Null}(-I - JH^N) \leq n^2.$$

From (23), we obtain(17).

From (17), the singular value decomposition of $(I - JH^N)$ is given by

$$(I - JH^N) = U\begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix}V^* ,$$

where U, V are unitary matrices, and $\Sigma_1 \in R^{n^2 \times n^2}$ is a diagonal matrix whose diagonal elements are nonzero singular values of $(I - JH^N)$. Thus the constraint on an $X(0)$ and $X(N)$ pair satisfying (10) is given by

$$[R_1 \quad R_2]\begin{bmatrix} csX(0) \\ csX(N) \end{bmatrix} = 0, \tag{25}$$

where

$$[R_1 \quad R_2] \equiv [\Sigma_1 \quad 0]V^*.$$

Using (25), the coupled equations (9) can be reduced to a simple linear equation.

Lemma 6: An $X(0)$ and $X(N)$ pair satisfying (9) can be computed by the following equation:

$$\begin{bmatrix} (A_0{}'\otimes A_0{}') + (A_1{}'\otimes A_1{}') - I & (A_0{}'\otimes A_1{}')T + (A_1{}'\otimes A_0{}') \\ R_1 & R_2 \end{bmatrix}\begin{bmatrix} csX(0) \\ csX(N) \end{bmatrix} = \begin{bmatrix} -csQ \\ 0 \end{bmatrix}, \tag{26}$$

where R_1 and R_2 are from (25).

Proof: The upper equation of (26) is from the first equation of (9) and the lower equation is from (25). □

Remark 1: Once $X(0)$ and $X(N)$ are obtained, $X(i), 2 \le i \le N-1$ can be computed easily from (10). For example, $X(1)$ and $X(N-1)$ are computed by

$$\begin{bmatrix} csX(1) \\ csX(N-1) \end{bmatrix} = H \begin{bmatrix} csX(0) \\ csX(N) \end{bmatrix}.$$

Hence the Lyapunov equation (3) can be solved from a simple linear equation (26) and up to N-th power of the constant matrix H.

3. STABILITY CONDITION

In this section, we show that eigenvalues and eigenvectors of H are closely connected with frequency domain stability of (1). Based on this observation, we propose a new stability condition which ensure stability of (1) for all $N \in [0, N_{\max}]$.

System (1) is stable if and only if the characteristic equation

$$\det(z^{N+1}I - A_0 z^N - A_1) = 0 \tag{27}$$

has all its roots inside the unit circle. Since $\det M = \det M'$, (1) is stable if and only if the characteristic equation

$$\det(z^{N+1}I - A_0{}' z^N - A_1{}') = 0 \tag{28}$$

has all its roots inside the unit circle.

We will provide a simple way to check whether all roots of (28) lie inside the unit circle. Define

$$\begin{aligned} &f(z,r) \equiv \det(z^{r+1}I - A_0{}' z^r - A_1{}'), \\ &W(r) \equiv \{z \in C \mid f(z,r) = 0\}, \\ &W_B(r,\varepsilon) \equiv \{z \in C \,\|\, z - \hat{z} < \varepsilon, \hat{z} \in W(r)\}. \end{aligned} \tag{29}$$

In the next lemma, we show that if r is changed to $r + \Delta r$ and Δr is small, each element of $W(r+\Delta r)$ is near $W(r)$.

Lemma 7: For any $\varepsilon > 0$, there is $\delta > 0$ such that for every $z \in W(r+\Delta r)$, the following is satisfied:

$$z \in W_B(r,\varepsilon) \text{ whenever } |\Delta r| < \delta. \tag{30}$$

Proof: Note that $f(z,r)$ can be written as follows:

$$f(z,r) = \sum_{k=0}^{n} a_k(z) z^{-kr}$$

for some polynomials $a_k(z)$ and $f(z,r)$ is an analytic function of z. Thus $|f(z,r)|$ is a continuous function of z. If we define

$$m(r,\varepsilon) \equiv \inf_{z\in C - W_B(r,\varepsilon)} |f(z,r)|, \tag{31}$$

then

$$z \in W_B(r,\varepsilon) \text{ whenever } |f(z,r)| < m(r,\varepsilon). \tag{32}$$

Now suppose $z \in W(r+\Delta r)$. Since $f(z,r)$ is an analytic function of r, for any $m(r,\varepsilon) > 0$, there is $\delta > 0$ such that

$$|f(z,r+\Delta r) - f(z,r)| < m(r,\varepsilon) \text{ whenever } |\Delta r| < \delta. \tag{33}$$

Since $f(z, r+\Delta r) = 0$, we have

$$|f(z,r)| < m(r,\varepsilon) \text{ whenever } |\Delta r| < \delta.$$

From (32), this implies (30). □

Lemma 7 states that small change of r guarantees small change of the location of $W(r)$. From this observation, we obtain the following lemma.

Lemma 8: If (1) is stable for $N = 0$ and $W(r)$ does not have an element on the unit circle for all real number $r \in [0, N_{\max}]$, then (1) is stable for $N \in [0, N_{\max}]$.

Proof: Let $\rho(r)$ be defined by

$$\rho(r) \equiv \sup_{z\in W(r)} |z|.$$

From Lemma 7, for each $\varepsilon > 0$, there is $\delta > 0$ such that

$$\rho(r+\Delta r) < \rho(r) + \varepsilon \quad \text{whenever} \quad |\Delta r| < \delta .$$

Thus if $\rho(r) < 0$ and $z \in W(r)$ is not on the unit circle for $0 \le r \le N_{\max}$, $\rho(r) < 1$ for $0 \le r \le N_{\max}$. This means that all roots of (28) lies inside the unit circle for $0 \le N \le N_{\max}$. □

The next lemma shows that unit circle element of $W(r)$ can be checked from eigenvalues of H.

Lemma 9: If $W(r)$ has a unit circle element, then the root is an eigenvalue of H.

Proof: If $f(z,r) = 0$ has a root $z = e^{jw}, w \in R$ on the unit circle, there is a nonzero $x \in C^n$ satisfying

$$(e^{jw} I - A_0' - A_1' e^{-jwr}) x = 0. \tag{34}$$

Let $\alpha \in C^n$ be defined by

$$\alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = x e^{-\frac{jwr}{2}} .$$

Let $v \in C^{2n^2}$ be defined by

$$v = \begin{bmatrix} u \\ \overline{u} \end{bmatrix}, \tag{35}$$

where u is defined by

$$u \equiv \begin{bmatrix} \overline{\alpha}_1 x \\ \vdots \\ \overline{\alpha}_n x \end{bmatrix}.$$

Note $v \neq 0$ from the construction. We will prove this lemma by showing that $(e^{jw} I - H) v = 0$. From the definition of H, we obtain

$$\begin{bmatrix} p \\ q \end{bmatrix} \equiv (e^{jw} I - H) v = \begin{bmatrix} (e^{jw} I - A) u - B\overline{u} \\ A^{-1} B A u + (e^{jw} I - A^{-1}(I - BB))\overline{u} \end{bmatrix}.$$

We will show that $p = q = 0$. The i-th block of p is given by

$$\begin{aligned} p_i &= (e^{jw} I - A_0')\overline{\alpha}_i x - A_1' \alpha \overline{\alpha}_i e^{-\frac{jwr}{2}} \\ &= \overline{\alpha}_i (e^{jw} I - A_0' - A_1' e^{-jwr}) x = 0. \end{aligned} \tag{36}$$

The lower block vector q is given by

$$\begin{aligned} q &= A^{-1} B A u + (e^{jw} I - A^{-1}(I - BB))\overline{u} \\ &= A^{-1}\{BAu + (e^{jw} A - I + BB)\overline{u}\}. \end{aligned}$$

Using $Au = e^{jw} u - B\overline{u}$ (from the fact $p = 0$), we obtain:

$$\begin{aligned} q &= A^{-1}\{e^{jw} Bu - BB\overline{u} + (e^{jw} A - I + BB)\overline{u}\} \\ &= A^{-1} e^{jw}\{Bu + (A - e^{jw} I)\overline{u}\} = A^{-1} e^{jw}\{-\overline{p}\} = 0. \end{aligned}$$

The last equation is from (36). Thus we have proved $(e^{jw} I - H) v = 0$. □

Using Lemma (9), we can compute $N_{\max}$ such that (1) is stable for all $N \in [0, N_{\max}]$.

Theorem 1: Let $e^{jw_i}, w_i \in R \ge 0$ be a unit circle eigenvalue of H and v_i be the corresponding eigenvector. Let $r_i \in R \ge 0$ be defined by

$$r_i \equiv \begin{cases} |\operatorname{Im}(\ln(\frac{\beta_k}{\gamma_k}))| / w_i, & w_i \neq 0, \\ 0, & w_i = 0, \end{cases} \tag{37}$$

where β_k is k-th element of v_i, γ_k is $(n^2 + k)$-th element of v_i. $k \le n^2$ can be chosen arbitrarily as long as k-th element of v_i is nonzero.

(i) If (1) is stable for $N = 0$ and $N_{\max}$ is the greatest integer not larger than $\min r_i$, then (1) is stable for all $N \in [0, N_{\max}]$.

(ii) If (1) is stable for $N = 0$ and H does not have a unit circle eigenvalue, then (1) is stable for all $N \ge 0$.

Proof: Suppose $k = 1$ in (37). From (35), we obtain

$$\frac{\beta_1}{\gamma_1} = e^{jw_i r_i}$$

and $r = r_i$ is a unit circle root of $f(z,r) = 0$. From Lemma 9, $W(r)$ does not have an element on the unit circle for $r \le \min r_i$. Thus (28) does not have a root on the unit circle for $N \le \min r_i$. Hence from Lemma 8, (1) is stable for all $N \in [0, N_{\max}]$. The proof of (ii) is immediate from Lemma 8 and Lemma 9. □

Remark 2: The stability condition (ii) is called *delay independent stability condition* [5,6].

4. NUMERICAL EXAMPLE

Consider the following system

$$x(k+1) = \begin{bmatrix} 0.3 & 0.15 \\ 0 & 0.7 \end{bmatrix} x(k) + \begin{bmatrix} 0.1 & -0.2 \\ 0.1 & -0.4 \end{bmatrix} x(k-N). \tag{38}$$

The system is stable for $N = 0$. Eigenvalues of H are given by

{0.2919, 0.3012, 0.6826, 0.9721 ± j 0.2346, 1.4650, 3.3205, 3.4256}.

Note that there exists a unit circle root $0.9721 + j\,0.2346 = e^{jw}$, $w \geq 0$, and $w = 0.2368$. The corresponding eigenvector v is given by

$$v = [-0.0784, -0.2921, -0.1687, -0.6155, -0.0189, 0.1988, -0.0299, -0.678]'.$$

From (37), we obtain $r = 10.2483$, and thus $N_{\max} = 10$. Hence we can conclude that (38) is stable for $N \in [0,10]$. In fact, by checking A_N, we can verify that (38) is stable for $N \leq 10$ and unstable $N = 11$.

For comparison, the stability is also checked with the stability condition in [9]: the system is determined to be stable for $N \in [0, 7]$.

Also, to check delay independent stability condition, stability of (38) is checked with

$$A_1 = \alpha \begin{bmatrix} 0.1 & -0.2 \\ 0.1 & -0.4 \end{bmatrix}.$$

It turns out that the system is delay independently stable (i.e., H matrix does not have unit circle eigenvalues) if $0 \leq \alpha \leq 0.834$. For comparison, the delay independent stability is also checked with the stability condition in [4]: the system is determined to be delay independently stable if $0 \leq \alpha \leq 0.834$. To investigate how two delay independent stability conditions are related is an interesting further research issue.

4. CONCLUSION

The main contribution of this paper is as follows. First, we have proposed an easy method to solve a Lyapunov equation for a state delay system. Using the proposed method, we can easily solve a Lyapunov equation even for large N; thus the proposed method can be useful when a Lyapunov equation for large N should be solved, for example gramian computation in the balanced model reduction of a large delay state delay system. Secondly, we have proposed a new stability condition based on the relationship between frequency domain stability and a constant matrix that appears in a Lyapunov equation. The proposed stability condition ensures stability of a discrete state delay system whose state delay is not exactly known but known to lie in a certain interval.

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