A Robust Extended Filter Design for SDINS In-Flight Alignment

Myeong-Jong Yu and Sang Woo Lee

Abstract: In the case of a strapdown inertial navigation system (SDINS) with sizeable attitude errors, the uncertainty caused by linearization of the system degrades the performance of the filter. In this paper, a robust filter and various error models for the uncertainty are presented. The analytical characteristics of the proposed filter are also investigated. The results show that the filter does not require the statistical property of the system disturbance and that the region of the estimation error depends on a freedom parameter in the worst case. Then, the uncertainty of the SDINS is derived. Depending on the choice of the reference frame and the attitude error state, several error models are presented. Finally, various in-flight alignment methods are proposed by combining the robust filter with the error models. Simulation results demonstrate that the proposed filter effectively improves the performance.

Keywords: Robust filter, uncertainty, SDINS, quaternion error model.

1. INTRODUCTION

The strapdown inertial navigation system (SDINS) is a nonlinear system. Because of the inherited inertial sensor errors and initial navigation errors, the errors of navigation solutions have a tendency to increase with time. To solve these problems, an aided SDINS is designed with external sensors. In designing the aided SDINS, the filter design and the SDINS error model are the main concerns [1-4]. A considerable amount of effort has been devoted to developing effective error models. However, in the case of a system with significant attitude errors, it is difficult to accurately linearize the SDINS error model. Therefore, error models include significant parameter uncertainty. This uncertainty degrades performance of the filter [5-6]. During the last several decades, the Kalman filter and the extended Kalman filter have been widely used in the aided SDINS. Not just precise system modeling, but also the statistical property of the system disturbance is required for these filters. However, in reality, the model uncertainty and the incompleteness of the statistical information make it complicated to estimate the states of a system with any accuracy. These difficulties can be overcome by studying a robust filter [7-13]. For given statistical information and modeling uncertainty, the H_2 filter has most often been constructed to estimate the system errors. A guaranteed cost minimization has been widely used as the performance index of H_2 filter and upper bound minimization or minimum variance of the estimation errors are used as the cost [7-9]. However, these methods require the known statistical information. Robust filters for a class of uncertain systems have recently been presented in [14-15]. This approach is concerned with constructing a state estimator for a class of uncertain linear systems with an integral quadratic constraint. Although the presented filter structure is similar to that of a previous H_2 filter, it requires no exact statistical information concerning the noise. This approach shows that it is possible to derive a robust filter in spite of system noise subject to an L_2 norm. For a nonlinear system, a nonlinear robust filter with Hamilton-Jacobi inequality has been developed. However, it is computationally complex and also has strong restricted conditions for obtaining the filter gain. For these reasons, an approximated solution to the robust filtering problem has been developed based on a linearization method. This design method leads to a realistic filter formulation similar to the extended Kalman filter [10-11]. In [17], the nonlinear state estimation with a similar approach is proposed for a nonlinear uncertain system with uncertainties described by an integral quadratic constraint. In this paper, a robust filter for uncertainty of the SDINS is presented. The derivation is similar to that of [17]. The robust filter is constructed, similarly to the extended Kalman filter, with local linearization of the system at the reference point. By introducing a state

Manuscript received April 17, 2003; revised October 20, 2003; accepted November 4, 2003. Recommended by Editorial Board member Jae Weon Choi under the direction of Editor Keum-Shik Hong.

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estimation set that is the solution of the Hamilton-Jacobi-Bellman partial differential and by solving the filtering problem locally, a robust filter is derived. Then the characteristics of the filter are analyzed.

SDINS error models are obtained based on the assumption that the attitude errors are sufficiently large [5-6]. These address the uncertainties generated in the modeling errors that are essential in the robust filter. The robust filter requires that the system uncertainty can be converted into factorized matrix form. This paper also presents a method to convert the uncertainty into factorized form. The proposed filter is applied to the SDINS in-flight alignment.

2. DESIGN AND ANALYSIS OF THE ROBUST EXTENDED FILTER

Consider a nonlinear uncertain system described by

$$\dot{x}(t) = f(x(t)) + \Delta A(x(t)) + B_2(t)w_o(t), \quad (1)$$

$$y(t) = h(x(t)) + v(t), \quad (2)$$

where $\Delta A(x(t))$ represents the system uncertainty and has the factorized form

$$\Delta A(x(t)) = B_1(t)\Delta_1(t)N(x(t)) \quad (3)$$

where $B_1(t)$ and $N(x(t))$ are known matrices. $\Delta_1(t)$ is an unknown matrix satisfying the condition $\left\| Q_1^{-1/2}\Delta_1(t) \right\| \le 1$ where Q_1 is a bounded positive definite matrix. $w_o(t)$ is the process noise and $v(t)$ is the measurement noise. They belong to the set of $L_2[0,T]$ norm and the statistical properties are unknown. In addition, $w_o(t)$ and $v(t)$ are noises satisfying the bound

$$\Phi(x(0)) + \int_0^T [w_o^T(t)Q_2^{-1}w_o(t) + v^T(t)R^{-1}v(t)]dt \le d$$

where $0 \le t \le T$, $\Phi(x(0))$ is a bounded positive function that depends on an initial state and d is an assigned positive real number. Converting the uncertainty to a fictitious noise and introducing a free parameter, uncertain systems (1) and (2) can be transformed into an auxiliary system,

$$\dot{x}(t) = f(x(t)) + B(t)w(t), \quad (4)$$

$$y(t) = h(x(t)) + v(t), \quad (5)$$

where

$$B(t) = [\rho B_1(t) \quad B_2(t)], \quad (6)$$

$$\omega(t) = \begin{bmatrix} \rho^{-1}\Delta_1(t)N(x(t)) \\ w_o(t) \end{bmatrix}, \quad (7)$$

and ρ is the free parameter. To construct a robust filter, it is assumed that systems (4) and (5) satisfy Assumptions 1-5.

A1: Every function shown in (4) and (5) belongs to C^1 and the first derivative is bounded.

A2: The matrix $N(x(t))$ is bounded.

A3: The functions Φ, L_1, and L_2 belong to C^1 and are bounded nonnegative functions. They also satisfy

$$\|\Phi(x_2) - \Phi(x_1)\| \le \theta_1(1 + \|x_2\| + \|x_1\|)\|x_2 - x_1\|, \quad (8a)$$

$$\|L_1(x_2) - L_1(x_1)\| \le \theta_2(1 + \|x_2\| + \|x_1\|)\|x_2 - x_1\|, \quad (8b)$$

$$\|L_2(x_2) - L_2(x_1)\| \le \theta_3(1 + \|x_2\| + \|x_1\|)\|x_2 - x_1\|, \quad (8c)$$

where θ_1, θ_2, and $\theta_3 > 0$.

A4: The function L_1 satisfies a coercivity condition,

$$L_1(w,v) \ge c\|w\|^2 \text{ where } c > 0. \quad (9)$$

A5: The matrix B is of full rank.

2.1. Design of the robust extended filter

In this section, a robust filter is derived based on a local solution of the filter problem. Similar to the development of the well-known extended Kalman filter, we derive the filter by linearizing the system in a neighborhood of the estimated trajectory, $\hat{x}(t)$.

We consider the system that satisfies an integral quadratic constraint given by

$$(x(0) - x_o)^T M(x(0) - x_o) + \int_0^T L_1(w,v)dt \le d + \int_0^T L_2(N(x))dt, \quad (10)$$

where

$$L_1(w,v) = w^T Q^{-1} w + v^T R^{-1} v = w^T \begin{bmatrix} Q_1^{-1} & 0 \\ 0 & Q_2^{-1} \end{bmatrix} w + v^T R^{-1} v, \quad (11)$$

$$L_2(N(x)) = \rho^{-2} N(x)^T N(x), \quad (12)$$

and Q_2 is a bounded positive definite matrix [17, 18].

We define the function

$$V(x,t) = \min_w[(x(0)-x_0)^T M(x(0)-x_0) + \int_0^t [L_1(w(t),v(t)) - L_2(N(x(t)))]dt]. \tag{13}$$

For systems (4) and (5) combined with (10), a partial differential equation is generally given by

$$\frac{\partial}{\partial t}V(x,t) + \max_w[\nabla_x V(f(x)+Bw) - L_1(w,v)) + L_2(N(x))] = 0, \tag{14}$$

where $V(x,t)$ denotes a value function and $V(x,0)=\Phi(0)$. Assumptions A1-A5 ensure that $V(x,t)$ is finite [18].

To derive a robust filter with a modified H_2 filter structure, we consider a system that satisfies an integral quadratic constraint provided by

$$\begin{aligned}&(x(0)-x_0)^T M(x(0)-x_0) \\ &+\frac{1}{2}\int_0^T [w(t)^T Q^{-1} w(t) + v(t)^T R^{-1} v(t)]dt \\ &\le \quad d + \frac{1}{2}\int_0^T [\rho^{-2} N(x(t))^T N(x(t))]dt.\end{aligned} \tag{15}$$

Using (14), the partial differential equation for (15) is obtained as

$$\begin{aligned}&\frac{\partial}{\partial t}V(x,t) + \max_w[\nabla_x V(f(x)+Bw) \\ &-\frac{1}{2}[w(t)^T Q^{-1} w(t) + v(t)^T R^{-1} v(t)] \\ &+\frac{1}{2}[\rho^{-2} N(x(t))^T N(x(t))] = 0.\end{aligned} \tag{16}$$

Rearranging the term $[\nabla_x VBw - \frac{1}{2}w^T Q^{-1} w]$ of (16), we get

$$\begin{aligned}&\nabla_x VBw - \frac{1}{2}w^T Q^{-1} w = \frac{1}{2}\nabla_x VBQB^T \nabla_x V^T \\ &-\frac{1}{2}((\nabla_x VBQ^{\frac{1}{2}})^T - Q^{-\frac{1}{2}}w)^T((\nabla_x VBQ^{\frac{1}{2}})^T - Q^{-\frac{1}{2}}w).\end{aligned} \tag{17}$$

Using (17), the partial differential equation can be derived as

$$\begin{aligned}&\frac{\partial}{\partial t}V + \nabla_x Vf(x) + \frac{1}{2}\nabla_x VBQB^T \nabla_x V^T \\ &-\frac{1}{2}(y-h(x))^T R^{-1}(y-h(x)) + \frac{1}{2}\rho^{-2} N(x)^T N(x) = 0\end{aligned} \tag{18}$$

where $V(x(0),0) = (x(0)-x_0)^T M(x(0)-x_0)$. In the general nonlinear case, solutions of (18) are smooth and must be interpreted in the viscosity sense. Thus, a filter is not well defined in the large. Therefore, we seek for an approximate $V((x),t)$ by a quadratic form and replace (18) by a simpler Riccati equation [17, 18].

$\hat{x}(t)$, as an estimated value of the state variable $x(t)$, is defined to be

$$\hat{x}(t) = \arg\min_x V((x(t),t). \tag{19}$$

(19) satisfies two conditions:

$$\nabla_x V(\hat{x}(t),t) = 0, \tag{20}$$

$$\nabla_x^2 V(\hat{x}(t),t)\dot{\hat{x}}(t) + \frac{\partial}{\partial t}\nabla_x V(\hat{x}(t),t)^T = 0. \tag{21}$$

The gradient of (18) with respect to x is given by

$$\begin{aligned}&\frac{\partial}{\partial t}\nabla_x V^T + \nabla_x f(x)^T \nabla_x V^T + \nabla_x^2 VBQB^T \nabla_x V^T \\ &+\nabla_x^2 Vf(x) + \nabla_x h(x)^T R^{-1}(y-h(x)) \\ &+\rho^{-2}\nabla_x N(x)^T N(x) = 0.\end{aligned} \tag{22}$$

Using (20) and (21) and evaluating at $x=\hat{x}$, (22) is simplified as

$$\begin{aligned}&\nabla_x^2 V(\hat{x},t)\dot{\hat{x}} = \nabla_x^2 V(\hat{x},t)f(\hat{x}) \\ &+\nabla_x h(\hat{x})^T R^{-1}(y-h(\hat{x})) + \rho^{-2}\nabla_x N(\hat{x})^T N(\hat{x}).\end{aligned} \tag{23}$$

Furthermore, supposing that the matrix $\nabla_x^2 V(\hat{x},t)$ is nonsingular for all t, the dynamic equation of the state estimate satisfying (19) can be written as

$$\begin{aligned}\dot{\hat{x}}(t) = f(\hat{x}(t)) &+ (\nabla_x^2 V(\hat{x}(t),t)^{-1} \\ &[\nabla_x h(\hat{x}(t))^T R^{-1}(y-h(\hat{x}(t))) \\ &+\rho^{-2}\nabla_x N(\hat{x}(t))^T N(\hat{x}(t))].\end{aligned} \tag{24}$$

In addition, the gradient of (22) with respect to x is expressed as

$$\begin{aligned}&\frac{\partial}{\partial t}\nabla_x^2 V + \nabla_x f(x)^T \nabla_x^2 V^T + \nabla_x^2 f(x)^T \nabla_x V^T \\ &+\nabla_x^2 V\nabla_x f(x) + \nabla_x^3 Vf(x) + \nabla_x^3 VBQB^T \nabla_x V^T \\ &+\nabla_x^2 VBQB^T \nabla_x^2 V^T - \nabla_x h(x)^T R^{-1}\nabla_x h(x) \\ &+\nabla_x^2 h(x)^T R^{-1}(y-h(x)) + \rho^{-2}\nabla_x^2 N(x)^T N(x) \\ &+\rho^{-2}\nabla_x N(x)^T \nabla_x N(x) = 0.\end{aligned} \tag{25}$$

Because $V(x,t)$ is quadratic, the third-order gradient terms vanish. Using (20) and (21) and evaluating at $x = \hat{x}$, (25) is simplified as

$$\begin{aligned}&\dot{\Pi} + \nabla_x f(\hat{x})^T \Pi + \Pi \nabla_x f(\hat{x}) + \Pi BQB^T \Pi \\ &-\nabla_x h(\hat{x})^T R^{-1} \nabla_x h(\hat{x})) \\ &+\rho^{-2} \nabla_x N(\hat{x})^T \nabla_x N(\hat{x}) = 0\end{aligned} \tag{26}$$

where $\Pi = \nabla_x^2 V$ and $\Pi(0) = M$. Furthermore, the corresponding differential equation for $P(t) = \Pi(t)^{-1}$ from (26) is

$$\begin{aligned}\dot{P}(t) &= P(t)\nabla_x f(\hat{x})^T + \nabla_x f(\hat{x}) P(t) \\ &+ BQB^T - P(t)\nabla_x h(\hat{x})^T R^{-1} \nabla_x h(\hat{x}) P(t) \\ &+ \rho^{-2} P(t) \nabla_x N(\hat{x})^T \nabla_x N(\hat{x}) P(t)\end{aligned} \tag{27}$$

where $P(0) = M^{-1}$ and M is the matrix that reflects the initial errors of the system. From these results, a robust filter for systems (1) and (2) can be summarized as

$$\begin{aligned}\dot{\hat{x}}(t) &= f(\hat{x}(t)) + P(t)\nabla_x h(\hat{x}(t))^T R^{-1}(y - h(\hat{x}(t)) \\ &+\rho^{-2} P(t) \nabla_x N(\hat{x}(t))^T N(\hat{x}(t)),\end{aligned} \tag{28}$$

$$\begin{aligned}\dot{P}(t) &= P(t)\nabla_x f(\hat{x})^T + \nabla_x f(\hat{x}) P(t) + \rho^2 B_1 Q_1 B_1^T \\ &+ B_2 Q_2 B_2^T - P(t) \nabla_x h(\hat{x})^T R^{-1} \nabla_x h(\hat{x}) P(t) \\ &+\rho^{-2} P(t) \nabla_x N(\hat{x})^T \nabla_x N(\hat{x}) P(t)\end{aligned} \tag{29}$$

where $\hat{x}(0) = x_0$. The proposed filter shows a modified structure of the nonlinear H_2 filter presented in [17]. This filter is particularly developed by employing a free parameter ρ which is used as a tuning parameter. It has the advantage of being able to be turned to improve performance. The free parameter plays an important role in accomplishing superior performance when the robust filter is applied to the real system. In Section 3, it will be shown that this parameter can be effectively utilized to improve the performance of a SDINS in-flight alignment application.

2.2 Analysis of the robust extended filter

In this section, the analytical characteristics of the proposed filter are investigated. We will consider the state estimation set that represents the region of the estimation error in the worst case, as an important property of the filter.

Now, a state estimation set is derived. The estimation error can be defined as

$$\zeta(t) = x(t) - \hat{x}(t). \tag{30}$$

Neglecting higher-order terms of the estimation errors, nonlinear functions $f(x(t))$, $h(x(t))$, and $N(x(t))$ are defined to be

$$\begin{aligned}f(x(t)) &\cong f(\hat{x}(t)) + A(t)(x(t) - \hat{x}(t)), \\ h(x(t)) &\cong h(\hat{x}(t)) + C(t)(x(t) - \hat{x}(t)), \\ N(x(t)) &\cong N(\hat{x}(t)) + N(t)(x(t) - \hat{x}(t)),\end{aligned} \tag{31}$$

and the dynamic equation of the estimated error $\zeta(t)$ is expressed as

$$\begin{aligned}\dot{\zeta}(t) &= (A(t) - K(t)C(t))\zeta(t) + B(t)w(t) \\ &- \rho^{-2} P(t) \nabla_x N(\hat{x}(t))^T N(\hat{x}(t)) - K(t)v(t)\end{aligned} \tag{32}$$

where $A(t) = \dfrac{\partial f}{\partial x}(\hat{x}(t))$, $C(t) = \dfrac{\partial h}{\partial x}(\hat{x}(t))$, and $K(t) = P(t)C(t)^T R^{-1}$.

Suppose that a function is chosen such that

$$V(\zeta(t)) = \frac{1}{2}\zeta^T(t) P(t)^{-1} \zeta(t) \tag{33}$$

where $P(t)$ is the solution of (29). Differentiating $V(\zeta(t))$ over time yields

$$\begin{aligned}\dot{V}(\zeta(t)) &= \frac{1}{2}[\dot{\zeta}^T(t) P(t)^{-1} \zeta(t) \\ &+ \zeta^T(t) \dot{P}(t)^{-1} \zeta(t) + \zeta^T(t) P(t)^{-1} \dot{\zeta}(t)]\end{aligned} \tag{34}$$

Substituting (29) and (32) in (34), $\dot{V}(\zeta(t))$ becomes

$$\begin{aligned}&\dot{V}(\zeta(t)) = \frac{1}{2}[\zeta^T(t)[C(t)^{-1} R^{-1} C(t) \\ &-P(t)^{-1} B(t) Q B(t)^T P(t)^{-1} - (K(t)C(t))^T P(t)^{-1} \\ &-P(t)^{-1} K(t) C(t)]\zeta(t) + w^T B(t)^T P(t)^{-1} \zeta(t) \\ &-(K(t)v)^T P(t)^{-1} \zeta(t) + \zeta(t)^T P(t)^{-1} B(t) w \\ &-\zeta(t)^T P(t)^{-1} K(t) v \\ &+\rho^{-2}\{\zeta^T(t)[-\nabla_x N(\hat{x})^T \nabla_x N(\hat{x})]\zeta(t) \\ &-\zeta(t)^T \nabla_x N(\hat{x})^T N(\hat{x}) - (\nabla_x N(\hat{x})^T N(\hat{x}))^T \zeta(t)\}].\end{aligned} \tag{35}$$

Rearranging (35), we get

$$\begin{aligned}&\dot{V}(\zeta(t)) = \frac{1}{2}\{\zeta^T(t)[-P(t)^{-1} B(t) Q B(t)^T P(t)^{-1}]\zeta(t) \\ &+w^T B(t)^T P(t)^{-1} \zeta(t) + \zeta(t)^T P(t)^{-1} B(t) w\} \\ &+\frac{1}{2}\{\zeta^T(t)[-C(t)^T R^{-1} C(t)]\zeta(t) \\ &-(K(t)v)^T P(t)^{-1} \zeta(t) - \zeta(t)^T P(t)^{-1} K(t) v\} \\ &+\rho^{-2}\frac{1}{2}\{\zeta^T(t)[-\nabla_x N(\hat{x})^T \nabla_x N(\hat{x})]\zeta(t) \\ &-\zeta(t)^T \nabla_x N(\hat{x})^T N(\hat{x}) - (\nabla_x N(\hat{x})^T N(\hat{x}))^T \zeta(t)\}.\end{aligned} \tag{36}$$

The first term of the right hand side of (36) can be converted into

$$\begin{aligned}&\zeta(t)^T[-P(t)^{-1}B(t)QB(t)^T P(t)^{-1}]\zeta(t)\\&+w^T B(t)^T P(t)^{-1}\zeta(t)+\zeta(t)^T P(t)^{-1}B(t)w\\&=w^T Q^{-1}w-s^T s\end{aligned} \quad (37)$$

where $s=Q^{-\frac{1}{2}}w-(B(t)Q^{\frac{1}{2}})^T P(t)^{-1}\zeta$ and $Q=Q^{\frac{1}{2}}Q^{\frac{1}{2}T}$. The second term of the right hand side of (36) can be also converted into

$$\begin{aligned}&\zeta(t)^T[-C(t)^T R^{-1}C(t)]\zeta(t)-(K(t)v)^T P(t)^{-1}\zeta(t)\\&-\zeta(t)^T P(t)^{-1}K(t)v=v^T R^{-1}v-\eta^T R^{-1}\eta\end{aligned} \quad (38)$$

where $\eta=v+C(t)\zeta$ and $R=R^{\frac{1}{2}}R^{\frac{1}{2}T}$. The third term can be modified into

$$\begin{aligned}&\{\zeta^T(t)[-\nabla_x N(\hat{x})^T\nabla_x N(\hat{x})]\zeta(t)\\&-\zeta(t)^T\nabla_x N(\hat{x})^T N(\hat{x})-(\nabla_x N(\hat{x})^T N(\hat{x}))^T\zeta(t)\}\\&=N(\hat{x})^T N(\hat{x})-(N(\hat{x})+\nabla_x N(\hat{x})\zeta(t))^T(N(\hat{x})\\&+\nabla_x N(\hat{x})\zeta(t))=N(\hat{x})^T N(\hat{x})-N(x)^T N(x).\end{aligned} \quad (39)$$

By the substitution of (37), (38), and (39) into (36), we are left with

$$\begin{aligned}\dot{V}(\zeta(t))&=\frac{1}{2}[w^T Q^{-1}w-s^T s+v^T R^{-1}v-\eta^T R^{-1}\eta\\&\quad+\rho^{-2}\{N(\hat{x})^T N(\hat{x})-N(x)^T N(x)\}]\\&\le\frac{1}{2}[w_0^T Q_2^{-1}w_0-s^T s+v^T R^{-1}v-\eta^T R^{-1}\eta\\&\quad+\rho^{-2}N(\hat{x})^T N(\hat{x})].\end{aligned} \quad (40)$$

From (40), we obtain the following inequality:

$$\begin{aligned}\dot{V}(\zeta(t)) \;\le\; &\frac{1}{2}[w_0^T Q_2^{-1}w_0+v^T R^{-1}v\\&+\rho^{-2}N(\hat{x})^T N(\hat{x})].\end{aligned} \quad (41)$$

Finally, the approximated state estimation set, χ_{set}, of the derived filter is obtained as follows. By integrating both sides of (41) and by simplifying the results, the approximated state estimation set is given by

$$\begin{aligned}\chi_{set}&=\{x(T)\in R^n:\\&\quad\frac{1}{2}(x(T)-\hat{x}(T))^T P(T)^T(x(T)-\hat{x}(T))\\&\quad\le\; d+\frac{1}{2\rho^2}\|N(\hat{x})\|_2^2\}.\end{aligned} \quad (42)$$

The derived state estimation set is dependent upon ρ, d, and $N(\hat{x})$. This result shows that the free parameter, ρ, can be also utilized to reduce the filter's estimation set in the worst case. On the contrary, when the existing nonlinear H_2 filter is applied to the nonlinear system, the state estimation set depends on d and $N(\hat{x})$.

3. APPLICATION TO SDINS

In this section, error models including the uncertainty of the SDINS are developed. To verify the effectiveness of the derived uncertainty model, a velocity-aided SDINS is designed. The inertial navigation system is constructed in a local-level frame. The error models, such as latitude error δL, longitude error δl, height error δh, velocity error δv^n, and attitude error are adopted. The measurement of the velocity-aided SDINS is given by

$$y(t)=C(t)x(t)+v(t) \quad (43)$$

where $C(t)=[0_{3\times3}\;\; I_{3\times3}\;\; 0_{3\times9}]$. To employ the proposed robust filter, error models containing the uncertainty are necessary. Therefore, when the error models are derived, it is desirable to derive uncertainty structure. Depending on the choice of the reference frame and the attitude error state, several error models are derived. The position and velocity error models are derived as

$$\delta\dot{L}=\frac{R_{mm}\rho_E}{R_m+h}\delta L+\frac{\rho_E}{R_m+h}\delta h+\frac{1}{R_m+h}\delta v_N, \quad (44)$$

$$\begin{aligned}\delta\dot{l}&=\frac{\rho_N}{\cos L}(\tan L-\frac{R_{tt}}{R_t+h})\delta L\\&\quad-\frac{\rho_N\sec L}{R_t+h}\delta h+\frac{\sec L}{R_t+h}\delta v_E,\end{aligned} \quad (45)$$

$$\delta\dot{h}=-\delta v_D, \quad (46)$$

$$\begin{aligned}\delta\dot{v}^n&=\Delta C_b^n f^b-[2\omega_{ie}^n+\omega_{en}^n]\times\delta v^n+C_b^n\delta f^b\\&\quad+v^n\times(2\delta\omega_{ie}^n+\delta\omega_{en}^n)+\delta g^n.\end{aligned} \quad (47)$$

In the velocity error model, $\Delta C_b^n f^b$ is strongly related to attitude errors. In the case in which the system has only large attitude errors, this term contains uncertainty. To obtain effectively the uncertainty structure, we use a MQE (Multiplicative Quaternion Error) [5, 6, 13]. First, a MQE model with respect to the navigation frame is derived. The attitude error model, q^n, and $\Delta C_b^n f^b$ with respect to MQE are

$$\dot{q}^n = -\omega_{in}^n \times q^n - \frac{1}{2}(C_b^n \delta\omega_{ib}^b - \delta\omega_{in}^n), \quad (48)$$

$$\begin{aligned}\Delta C_b^n f^b &= -2[q_{0n}I + R_{Qn}][R_{Qn}]C_b^n f^b \\ &= 2[q_{0n}I + R_{Qn}][C_b^n f^b \times] q^n \\ &= 2[C_b^n f^b \times]q^n + 2[c_n I + R_{Qn}][C_b^n f^b \times] q^n\end{aligned} \quad (49)$$

where $q^n = [q_{1n} \quad q_{2n} \quad q_{3n}]^T$, $q_{0n} = 1 + c_n$ and $[R_{Qn}]$ denotes a skew-symmetric matrix of q^n. Then, the MQE model with respect to the body frame is obtained as follows. The attitude error model of the MQE, q^b, and $\Delta C_b^n f^b$ with respect to the body frame is

$$\dot{q}^b = -\omega_{in}^b \times q^b - \frac{1}{2}(\delta\omega_{ib}^b - C_n^b \delta\omega_{in}^n), \quad (50)$$

$$\begin{aligned}\Delta C_b^n f^b &= -2C_b^n[q_{0b}I + R_{Qb}][R_{Qb}] f^b \\ &= 2C_b^n[q_{0b}I + R_{Qb}][f^b \times] q^b \\ &= 2C_b^n[f^b \times]q^b + 2C_b^n[c_b I + R_{Qb}][f^b \times] q^b\end{aligned} \quad (51)$$

where $q^b = [q_{1b} \quad q_{2b} \quad q_{3b}]^T$, $q_{0b} = 1 + c_b$, and $[R_{Qb}]$ denotes a skew-symmetric matrix of q^b. (49) and (51) indicate that $\Delta C_b^n f^b$ is composed of the linear term and the uncertainty. Thus, the system equations involve uncertainty. Using the derived error models, we can easily convert the uncertainty into a factorized form, as $B_1(t)\Delta_1(t)N(x(t))$ in (3). Until now, the error models useful for the robust filter with the uncertainty structure are derived. By combining the robust filter proposed in Section 2 and derived error models, various in-flight alignment techniques can be designed. In this paper, we select the MQE model given by (44) - (49) for simulations. 15 state variables are composed of position (3), velocity (3), attitude (3), accelerometer bias (3), and gyroscope bias (3). The major errors of the inertial sensors considered in the simulation are gyroscope bias error (3 deg/hr), gyroscope scale factor error (500 ppm), and accelerometer bias error (1000 μg), with accelerometer bias assumed to be a random constant. The initial attitude errors are assumed to be 10 degrees horizontal plane attitude error and 20 degrees heading error. Monte Carlo simulations are performed for 1000 seconds and the filter update period is 0.1 second. The trajectory for the simulation is as follows. First, to increase the observability of the system, the heading angle is changed three times. After that time on, the vehicle moves straight with 10 m/sec speed. The free

Table 1. Steady-state position and heading errors.

	EKF	Robust filter
Heading error (arcmin)	80	50
Position error (C.E.P.)(m)	90	75

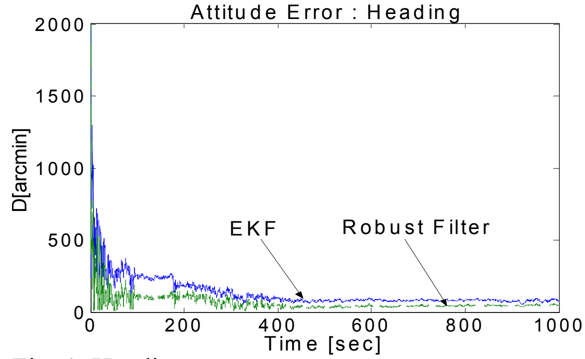

Fig. 1. Heading error.

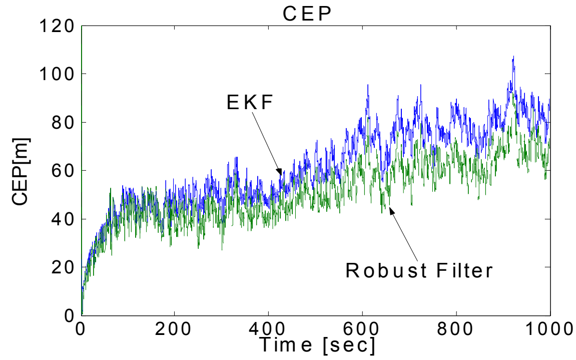

Fig. 2. Position error.

parameter ρ is selected as 0.5556 and is determined via simulations. The results are compared with the results of the corresponding EKF. The results are shown in Fig. 1, Fig. 2, and Table 1. The proposed filter reduces the attitude error and position error compared with the EKF. As shown in the figures, the proposed filter reduces the heading error by about 37% and the position error by about 16% more than the EKF. In addition, the convergence of the attitude error is faster than that of the EKF. The simulation results have shown that it is possible to further improve in-flight alignment accuracy by employing the proposed filter.

4. CONCLUSION

The robust filter suitable for the aided SDINS with substantial attitude errors is presented and the characteristics of the filter are analyzed. Then we obtain various error models, which can be utilized to improve the performance of the SDINS by employing the presented robust filter. The simulation results for a velocity-aided SDINS in-flight alignment have demonstrated that the proposed filter is more effective in estimating the attitude error and position error than the EKF.

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