H_∞ Control of 2-D Discrete State Delay Systems

Jianming Xu and Li Yu*

Abstract: This paper is concerned with the H_∞ control problem of 2-D discrete state delay systems described by the Roesser model. The condition for the system to have a specified H_∞ performance is derived via the linear matrix inequality (LMI) approach. Furthermore, a design procedure for H_∞ state feedback controllers is given by solving a certain LMI. The design problem of optimal H_∞ controllers is formulated as a convex optimization problem, which can be solved by existing convex optimization techniques. Simulation results are presented to illustrate the effectiveness of the proposed results.

Keywords: 2-D discrete systems, H_∞ control, LMI, state delay.

1. INTRODUCTION

Over the past several decades, two-dimensional (2-D) systems have received much interest due to their extensive applications in several modern engineering fields such as process control, image enhancement, image deblurring, signal processing, etc. [1-3]. 2-D state-space theory originated from Givone and Roesser [4,5] who proposed the celebrated Roesser model in the seventies of the 20th century. Since then, other scholars have drawn several different state-space models from their own research fields [6,7], such as FM LSS model. A great number of fundamental results on one-dimensional (1-D) systems have been extended to 2-D systems [1,8]. H_∞ control for 1-D systems has been one of most active research areas of control systems for the last two decades [9,10]. A main advantage of H_∞ control is that its performance specification takes account of the worst-case performance for system in terms of the system energy gain. This is appropriate for system robustness analysis and robust control with modeling uncertainties and disturbances than other performance specifications [11], such as the LQ-optimal control specification. The H_∞ control problem for 2-D systems was first addressed in [12]. Du and Xie established several versions of 2-D bounded real lemma [13].

On the other hand, time-delay phenomenon often appears in various engineering systems such as aircraft, chemical processes and networked control systems. It has been shown that the existences of delays in a dynamic system may result in instability, oscillations or performance deteriorated [14]. Therefore, the analysis and synthesis of 1-D time-delay systems has received a great deal of attention and has been one of the most interesting topics in the control over the decades [15-17]. Similarly, time-delay is often encountered in 2-D systems. However, few results have reported in literature on 2-D time-delay systems. Paszke *et al.* presented a sufficient stability condition and a stabilization method for discrete linear state delay 2-D systems with FM LSS model [18]. To the authors' knowledge, the H_∞ control problem for 2-D state delay systems has not been investigated. We extend the bounded real lemma for 2-D systems [13] to 2-D state delay systems and develop a design procedure for H_∞ state feedback controllers via the LMI approach.

In this paper, we are concerned with the H_∞ control problem of 2-D state delay systems described by the Roessor model. A sufficient condition for such a system to have a specified H_∞ performance is first presented via the LMI approach. Then a design procedure for H_∞ state feedback controllers is given by solving a certain LMI. Finally, for a class of 2-D discrete state delay systems with norm-bounded time-varying parameter uncertainties, the robust optimal state feedback H_∞ controller is obtained using convex optimization techniques.

2. H_∞ PERFORMANCE ANALYSIS OF 2-D DISCRETE STATE DELAY SYSTEMS

Consider the following 2-D discrete state delay system in the Roesser model:

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$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = A\begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + A_{d_1}x^h(i-d_1,j) + A_{d_2}x^v(i,j-d_2) + B_1 w(i,j) + B_2 u(i,j),$$
$$z(i,j) = H\begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + L_1 w(i,j) + L_2 u(i,j), \quad (1)$$

where i and j denote integer-valued horizontal and vertical coordinates, respectively, $x^h(i,j)\in \mathbf{R}^{n_1}$, $x^v(i,j)\in \mathbf{R}^{n_2}$, $u(i,j)\in \mathbf{R}^m$ and $z(i,j)\in \mathbf{R}^p$ denote, respectively, the horizontal state, the vertical state, the control input and the controlled output, $w(i,j)\in \mathbf{R}^q$ is the disturbance input which belongs to $\ell_2\{[0,\infty),[0,\infty)\}$, d_1 and d_2 are constant positive integers representing delays along horizontal direction and vertical direction, respectively. A, A_{d_1}, A_{d_2}, B_1, B_2, H, L_1 and L_2 are constant matrices with appropriate dimensions. The initial condition is defined as follows:

$$\begin{aligned} X(0) = \Big[& x^{h^{\mathrm T}}(-d_1,0),\ x^{h^{\mathrm T}}(-d_1,1),\ x^{h^{\mathrm T}}(-d_1,2),\ \cdots \\ & x^{h^{\mathrm T}}(1-d_1,0),\ x^{h^{\mathrm T}}(1-d_1,1),\ x^{h^{\mathrm T}}(1-d_1,2),\ \cdots \\ & x^{h^{\mathrm T}}(0,0),\ x^{h^{\mathrm T}}(0,1),\ x^{h^{\mathrm T}}(0,2),\ \cdots \\ & x^{v^{\mathrm T}}(0,-d_2),\ x^{v^{\mathrm T}}(1,-d_2),\ x^{v^{\mathrm T}}(2,-d_2),\ \cdots \\ & x^{v^{\mathrm T}}(0,1-d_2),\ x^{v^{\mathrm T}}(1,1-d_2),\ x^{v^{\mathrm T}}(2,1-d_2),\ \cdots \\ & x^{v^{\mathrm T}}(0,0),\ x^{v^{\mathrm T}}(1,0),\ x^{v^{\mathrm T}}(2,0),\ \cdots \Big]. \end{aligned} \quad (2)$$

For the system (1), assume a finite set of initial condition, i.e., there exist positive integers L and M, such that

$$\begin{aligned} & x^h(i,j)=0,\ \forall j\ge M,\ i=-d_1,\ -d_1+1,\cdots,\ 0, \\ & x^v(i,j)=0,\ \forall i\ge L,\ j=-d_2,\ -d_2+1,\cdots,\ 0. \end{aligned} \quad (3)$$

Denote $x^{\mathrm T}(i,j)=[x^{h^{\mathrm T}}(i,j)\quad x^{v^{\mathrm T}}(i,j)]$ and $X_r = \sup\{\|x(i,j)\| : i+j=r\}$, we first give the definition of asymptotic stability for the system (1).

Definition 1: The 2-D discrete state delay system (1) is asymptotically stable if $\lim_{r\to\infty} X_r = 0$ with zero input $u(i,j)=0$ and the initial condition (3).

Definition 2: Consider 2-D discrete state delay system (1) with the initial condition (3). Given a scalar $\gamma>0$ and symmetric positive definite weighting matrices R_h, R_v, S_h and S_v, the 2-D state delay system (1) with zero input $u(i,j)=0$ is said to have an H_∞ performance γ if it is asymptotically stable and satisfies

$$J = \sup_{0\ne(w,X(0))\in\ell_2} \frac{\|z\|_2^2}{\|w\|_2^2 + D_1(d_1,j)+D_2(i,d_2)} < \gamma^2, \quad (4)$$

where

$$D_1(d_1,j) = \sum_{j=0}^{\infty}\left[x^{h^{\mathrm T}}(0,j)R_h x^h(0,j) + \sum_{i=-d_1}^{-1} x^{h^{\mathrm T}}(i,j)S_h x^h(i,j)\right],$$
$$D_2(i,d_2) = \sum_{i=0}^{\infty}\left[x^{v^{\mathrm T}}(i,0)R_v x^v(i,0) + \sum_{j=-d_2}^{-1} x^{v^{\mathrm T}}(i,j)S_v x^v(i,j)\right].$$

In the case when the initial condition is known to be zero, i.e., $X(0)=0$, then the H_∞ performance measure (4) reduces to

$$J_0 = \sup_{0\ne w\in\ell_2} \frac{\|z\|_2}{\|w\|_2} < \gamma. \quad (5)$$

It follows from that the 2-D Parseval's theorem [3] that (5) is equivalent to

$$\|G(z_1,z_2)\|_\infty = \sup_{\omega_1,\,\omega_2\in[0,\,2\pi]} \sigma_{\max}[G(e^{j\omega_1},e^{j\omega_2})] < \gamma, \quad (6)$$

where $\sigma_{\max}(\cdot)$ denotes the maximum singular value of the corresponding matrix, and

$$\begin{aligned} G(z_1,z_2) = H(&\mathrm{diag}\{z_1 I_{n_1},\ z_2 I_{n_2}\} - A \\ & -[A_{d_1}z_1^{-d_1}I_{n_1}\quad A_{d_2}z_2^{-d_2}I_{n_2}])^{-1}B_1 + L_1 \end{aligned} \quad (7)$$

is the transfer function from the disturbance input $w(i,j)$ to the controlled output $z(i,j)$ for the 2-D state delay system (1).

The following theorem presents a sufficient condition for system (1) to have a specified H_∞ performance.

Theorem 1: Given a positive scalar γ, the 2-D state delay system (1) with the initial condition (3) has an H_∞ performance γ if there exist symmetric positive definite matrices $P=\mathrm{diag}\{P_h,\ P_v\}$ and $Q=\mathrm{diag}\{Q_h,\ Q_v\}$, where $P_h,\ Q_h\in\mathbf{R}^{n_1\times n_1}$ and $P_v,\ Q_v\in\mathbf{R}^{n_2\times n_2}$ satisfy $P_h<\gamma^2 R_h$, $P_v<\gamma^2 R_v$, $Q_h<\gamma^2 S_h$, and $Q_v<\gamma^2 S_v$, such that

$$\begin{bmatrix} A^{\mathrm{T}} \\ A_{d_1}^{\mathrm{T}} \\ A_{d_2}^{\mathrm{T}} \\ B_1^{\mathrm{T}} \end{bmatrix} P \begin{bmatrix} A & A_{d_1} & A_{d_2} & B_1 \end{bmatrix} + \begin{bmatrix} -P+Q+H^{\mathrm{T}}H & 0 & 0 & H^{\mathrm{T}}L_1 \\ 0 & -Q_h & 0 & 0 \\ 0 & 0 & -Q_v & 0 \\ L_1^{\mathrm{T}}H & 0 & 0 & L_1^{\mathrm{T}}L_1 - \gamma^2 I \end{bmatrix} < 0. \tag{8}$$

Proof: Suppose now that there exist symmetric positive define matrices $P = \operatorname{diag}\{P_h, P_v\}$ and $Q = \operatorname{diag}\{Q_h, Q_v\}$, such that the matrix inequality (8) holds. We denote the system state as

$$x(i,j) = \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix}, \; x'(i,j) = \begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix}, \tag{9}$$

and choose a Lyapunov functional

$$V(x(i,j)) = V_h(x^h(i,j)) + V_v(x^v(i,j)), \tag{10}$$

where

$$V_h(x^h(i,j)) = x^{h^{\mathrm{T}}}(i,j)P_h x^h(i,j) + \sum_{\theta=1}^{d_1} x^{h^{\mathrm{T}}}(i-\theta,j)Q_h x^h(i-\theta,j),$$

$$V_v(x^v(i,j)) = x^{v^{\mathrm{T}}}(i,j)P_v x^v(i,j) + \sum_{\theta=1}^{d_2} x^{v^{\mathrm{T}}}(i,j-\theta)Q_v x^v(i,j-\theta).$$

It is clear that $V(x(i,j))$ is positive.

The forward difference along any trajectory of the system (1) with $u(i,j)=0$ and $w(i,j)=0$ is given by

$$\begin{aligned}
&\Delta V(x(i,j)) \\
&= V(x'(i,j)) - V(x(i,j)) \\
&= x'^{\mathrm{T}}(i,j)Px'(i,j) - x^{\mathrm{T}}(i,j)Px(i,j) \\
&\quad + \sum_{\theta=1}^{d_1} x^{h^{\mathrm{T}}}(i+1-\theta,j)Q_h x^h(i+1-\theta,j) \\
&\quad - \sum_{\theta=1}^{d_1} x^{h^{\mathrm{T}}}(i-\theta,j)Q_h x^h(i-\theta,j) \\
&\quad + \sum_{\theta=1}^{d_2} x^{v^{\mathrm{T}}}(i,j+1-\theta)Q_v x^v(i,j+1-\theta) \\
&\quad - \sum_{\theta=1}^{d_2} x^{v^{\mathrm{T}}}(i,j-\theta)Q_v x^v(i,j-\theta) \\
&= \begin{bmatrix} x(i,j) \\ x^h(i-d_1,j) \\ x^v(i,j-d_2) \end{bmatrix}^{\mathrm{T}} \left(\begin{bmatrix} A^{\mathrm{T}} \\ A_{d_1}^{\mathrm{T}} \\ A_{d_2}^{\mathrm{T}} \end{bmatrix} P \begin{bmatrix} A & A_{d_1} & A_{d_2} \end{bmatrix} \right. \\
&\quad \left. + \begin{bmatrix} -P+Q & 0 & 0 \\ 0 & -Q_h & 0 \\ 0 & 0 & -Q_v \end{bmatrix} \right) \begin{bmatrix} x(i,j) \\ x^h(i-d_1,j) \\ x^v(i,j-d_2) \end{bmatrix}.
\end{aligned} \tag{11}$$

It follows from (8) that

$$\begin{bmatrix} A^{\mathrm{T}} \\ A_{d_1}^{\mathrm{T}} \\ A_{d_2}^{\mathrm{T}} \end{bmatrix} P \begin{bmatrix} A & A_{d_1} & A_{d_2} \end{bmatrix} + \begin{bmatrix} -P+Q & 0 & 0 \\ 0 & -Q_h & 0 \\ 0 & 0 & -Q_v \end{bmatrix} < 0.$$

This implies $\Delta V(x(i,j)) < 0$, i.e.,

$$\begin{aligned} &V_h(x^h(i+1,j)) + V_v(x^v(i,j+1)) \\ &< V_h(x^h(i,j)) + V_v(x^v(i,j)) \end{aligned} \tag{12}$$

for all $x(i,j) \neq 0$.

Let $D(r)$ denotes the set defined by

$$D(r) := \{(i,j) : i+j=r, \; i \geq 0, \; j \geq 0\}.$$

For any integer $r \geq \max\{L, M\}$, it follows from (12) and the initial condition (3) that

$$\begin{aligned}
&\sum_{(i+j)\in D(r)} V(x(i,j)) \\
&= V_h(x^h(r,\,0)) + V_h(x^h(r-1,\,1)) + V_h(x^h(r-2,\,2)) \\
&\quad + \cdots + V_h(x^h(1,\,r-1)) + V_h(x^h(0,\,r)) \\
&\quad + V_v(x^v(r,\,0)) + V_v(x^v(r-1,\,1)) + V_v(x^v(r-2,\,2)) \\
&\quad + \cdots + V_v(x^v(1,\,r-1)) + V_v(x^v(0,\,r)) \\
&\geq V_h(x^h(r+1,\,0)) + V_h(x^h(r,\,1)) + V_h(x^h(r-1,\,2)) \\
&\quad + \cdots + V_h(x^h(2,\,r-1)) + V_h(x^h(1,\,r)) \\
&\quad + V_v(x^v(r,\,1)) + V_v(x^v(r-1,\,2)) + V_v(x^v(r-2,\,3)) \\
&\quad + \cdots + V_v(x^v(1,\,r)) + V_v(x^v(0,\,r+1)) \\
&= \sum_{(i+j)\in D(r+1)} [V_h(x^h(i,\,j)) + V_v(x^v(i,\,j))] \\
&\quad - V_h(x^h(0,\,r+1)) - V_v(x^v(r+1,\,0)) \\
&= \sum_{(i+j)\in D(r+1)} V(x(i,\,j)),
\end{aligned} \tag{13}$$

where the equality sign holds only when

$$\sum_{(i+j)\in D(r)} V(x(i,\,j)) = 0.$$

This implies that the whole energies stored at the points $\{(i,j): i+j=r+1\}$ is strictly less than those at the points $\{(i,j): i+j=r\}$ unless all $x(i,j)=0$. Thus, we obtain

$$\lim_{r\to\infty}\sum_{(i+j)\in D(r)} V(x(i,\ j))=0. \tag{14}$$

It follows that

$$\lim_{i+j\to\infty} V(x(i,\ j))=0,\ \lim_{i+j\to\infty}\left\|x(i,\ j)\right\|=0,$$

which implies from Definition 1 that the system (1) is asymptotically stable.

To establish the H_∞ performance of the system (1) with the control input $u(i,j)=0$ for $w(i,j)\in \ell_2\{[0,\ \infty],\ [0,\ \infty]\}$, we consider

$$\Delta V(x(i,\ j))+z^{\mathrm{T}}(i,\ j)z(i,\ j)-(1-\tau)\gamma^2 w^{\mathrm{T}}(i,\ j)w(i,\ j)$$
$$=\begin{bmatrix} x(i,j)\\ x^h(i-d_1,j)\\ x^v(i,j-d_2)\\ w(i,j)\end{bmatrix}^{\mathrm{T}}\left(\begin{bmatrix} A^{\mathrm{T}}\\ A_{d_1}^{\mathrm{T}}\\ A_{d_2}^{\mathrm{T}}\\ B_1^{\mathrm{T}}\end{bmatrix} P\begin{bmatrix} A & A_{d_1} & A_{d_2} & B_1\end{bmatrix}\right.$$
$$\left.+\begin{bmatrix} -P+Q+H^{\mathrm{T}}H & 0 & 0 & H^{\mathrm{T}}L_1\\ 0 & -Q_h & 0 & 0\\ 0 & 0 & -Q_v & 0\\ L_1^T H & 0 & 0 & L_1^T L_1-(1-\tau)\gamma^2 I\end{bmatrix}\right)$$
$$\times\begin{bmatrix} x(i,j)\\ x^h(i-d_1,j)\\ x^v(i,j-d_2)\\ w(i,j)\end{bmatrix},$$

where τ is a positive scalar.

It follows from the inequality (8) that there always exists a positive scalar τ being small enough such that

$$\Delta V(x(i,j))+z^{\mathrm{T}}(i,j)z(i,j)-(1-\tau)\gamma^2 w^{\mathrm{T}}(i,j)w(i,j)<0.$$

Therefore, for any integers $p_1, p_2>0$, we have

$$\sum_{i=0}^{p_1}\sum_{j=0}^{p_2}\big[\Delta V(x(i,\ j))+z^{\mathrm{T}}(i,\ j)z(i,\ j) -\gamma^2 w^{\mathrm{T}}(i,\ j)w(i,\ j)\big]<0, \tag{15}$$

where

$$\sum_{i=0}^{p_1}\sum_{j=0}^{p_2}\Delta V(x(i,\ j))$$
$$=\sum_{i=0}^{p_1}\sum_{j=0}^{p_2}[V_h(x^h(i+1,j))+V_v(x^v(i,j+1)) -V_h(x^h(i,j))-V_v(x^v(i,j)]$$
$$=\sum_{j=0}^{p_2}[V_h(x^h(p_1+1,j))-V_h(x^h(0,j))] +\sum_{i=0}^{p_1}[V_v(x^v(i,p_2+1))-V_v(x^v(i,\ 0))]. \tag{16}$$

Let $p_2\ge p_1\ge \max\{L,\ M\}$, it follows from (12) and the initial condition (3) that

$$\sum_{j=0}^{p_2}V_h(x^h(p_1+1,j))$$
$$\le\sum_{j=0}^{p_2}[V_h(x^h(p_1,j))+V_v(x^v(p_1,j))-V_v(x^v(p_1,j+1))]$$
$$=V_h(x^h(p_1,\ 0))+V_v(x^v(p_1,\ 0))-V_v(x^v(p_1,p_2+1)) +\sum_{j=1}^{p_2}V_h(x^h(p_1,j))$$
$$\le V_h(x^h(p_1,\ 0))+V_v(x^v(p_1,\ 0))-V_v(x^v(p_1,p_2+1)) +\sum_{j=1}^{p_2}[V_h(x^h(p_1-1,j))+V_v(x^v(p_1-1,j)) -V_v(x^v(p_1-1,j+1))]$$
$$=V_h(x^h(p_1,\ 0))+V_v(x^v(p_1,\ 0))+V_h(x^h(p_1-1,\ 1)) +V_v(x^v(p_1-1,\ 1))-V_v(x^v(p_1,p_2+1)) -V_v(x^v(p_1-1,p_2+1))+\sum_{j=2}^{p_2}V_h(x^h(p_1-1,j))$$
$$\le\cdots\le\sum_{(i+j)\in D(p_1)}[V_h(x^h(i,\ j))+V_v(x^v(i,\ j))] +\sum_{j=p_1+1}^{p_2}V_h(x^h(0,\ j))-\sum_{i=0}^{p_1}V_v(x^v(i,p_2+1))$$
$$=\sum_{(i+j)\in D(p_1)}V(x(i,\ j))-\sum_{i=0}^{p_1}V_v(x^v(i,p_2+1)). \tag{17}$$

This implies

$$\sum_{j=0}^{p_2}V_h(x^h(p_1+1,j))+\sum_{i=0}^{p_1}V_v(x^v(i,p_2+1)) \le\sum_{(i+j)\in D(p_1)}V(x(i,\ j)). \tag{18}$$

Thus, when $p_2,\ p_1\to\infty$, it follows from (14)-(18) that

$$\|z\|_2^2 - \gamma^2 \|w\|_2^2 < \sum_{j=0}^{\infty} V_h(x^h(0,j)) + \sum_{i=0}^{\infty} V_v(x^v(i,\ 0))$$
$$= \sum_{j=0}^{\infty}[x^{h^{\mathrm T}}(0,j)P_h x^h(0,j) + \sum_{i=-d_1}^{-1} x^{h^{\mathrm T}}(i,j)Q_h x^h(i,j)]$$
$$+ \sum_{i=0}^{\infty}[x^{v^{\mathrm T}}(i,\ 0)P_v x^v(i,\ 0) + \sum_{j=-d_2}^{-1} x^{v^{\mathrm T}}(i,j)Q_v x^v(i,j)]. \tag{19}$$

Since $P_h < \gamma^2 R_h$, $P_v < \gamma^2 R_v$, $Q_h < \gamma^2 S_h$ and $Q_v < \gamma^2 S_v$, the inequality (19) leads to

$$\|z\|_2^2 < \gamma^2 \{\|w\|_2^2 + \sum_{j=0}^{\infty}[x^{h^{\mathrm T}}(0,j)R_h x^h(0,j) + \sum_{i=-d_1}^{-1} x^{h^{\mathrm T}}(i,j)S_h x^h(i,j)] \tag{20}$$
$$+ \sum_{i=0}^{\infty}[x^{v^{\mathrm T}}(i,0)R_v x^v(i,0) + \sum_{j=-d_2}^{-1} x^{v^{\mathrm T}}(i,j)S_v x^v(i,j)]\}.$$

Therefore, it follows from Definition 2 that system (1) has an H_∞ performance γ. This completes the proof. □

Remark 1: When the initial condition $X(0)$ is known to be zero, we need not present the weighting matrices R_h, R_v, S_h and S_v on zero boundary condition. Therefore, the requirements for $P_h < \gamma^2 R_h, P_v < \gamma^2 R_v$, $Q_h < \gamma^2 S_h$, and $Q_v < \gamma^2 S_v$ in Theorem 1 will become superfluous.

Remark 2: Theorem 1 provides a sufficient condition for the 2-D discrete state delay systems to be bounded real in terms of a certain LMI. For the 2-D system (1) without state delay, the LMI (8) reduces to

$$\begin{bmatrix} A^{\mathrm T} \\ B_1^{\mathrm T} \end{bmatrix} P \begin{bmatrix} A & B_1 \end{bmatrix} + \begin{bmatrix} -P + H^{\mathrm T}H & H^{\mathrm T}L_1 \\ L_1^{\mathrm T}H & L_1^{\mathrm T}L_1 - \gamma^2 I \end{bmatrix} < 0,$$

which is a sufficient condition for the 2-D systems to be bounded real in [13]. Therefore, Theorem 1 is an extension of bounded real lemma for 2-D discrete systems to 2-D state delay systems.

3. H_∞ CONTROLLER DESIGN OF 2-D DISCRETE STATE DELAY SYSTEMS

Consider the 2-D state delay system (1) and the following controller

$$u(i,j) = Kx(i,j). \tag{21}$$

The corresponding closed-loop system is given by

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = (A + B_2 K)\begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + A_{d_1} x^h(i-d_1,j) + A_{d_2} x^v(i,j-d_2) + B_1 w(i,j),$$
$$z(i,j) = (H + L_2 K)\begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + L_1 w(i,j). \tag{22}$$

If there exists the controller (21) such that the closed-loop system (22) is asymptotically stable, and the H_∞ norm of the transfer function (7) from the disturbance input $w(i, j)$ to the controlled output $z(i, j)$ for the system (22) is smaller than γ, then the closed-loop system (22) has a specified H_∞ performance γ, and the controller (21) is said to be a γ-suboptimal state feedback H_∞ controller for the 2-D state delay system (1).

Theorem 2: Consider the 2-D state delay system (1). Given a positive scalar γ, if there exist a matrix N and symmetric positive definite matrices $W=\mathrm{diag}\{W_h, W_v\}$ and $Y = \mathrm{diag}\{Y_h, Y_v\}$ such that

$$\begin{bmatrix} -W+Y & 0 & 0 & 0 & WA^{\mathrm T} + N^{\mathrm T}B_2^{\mathrm T} & WH^{\mathrm T} + N^{\mathrm T}L_2^{\mathrm T} \\ 0 & -Y_h & 0 & 0 & W_h A_{d_1}^{\mathrm T} & 0 \\ 0 & 0 & -Y_v & 0 & W_v A_{d_2}^{\mathrm T} & 0 \\ 0 & 0 & 0 & -\gamma^2 I & B_1^{\mathrm T} & L_1^{\mathrm T} \\ AW + B_2 N & A_{d_1}W_h & A_{d_2}W_v & B_1 & -W & 0 \\ HW + L_2 N & 0 & 0 & L_1 & 0 & -I \end{bmatrix} < 0. \tag{23}$$

Then the closed-loop system (22) has a specified H_∞ performance γ, and

$$u(i,j) = NW^{-1}x(i,j) \tag{24}$$

is a γ-suboptimal state feedback H_∞ controller for the 2-D state delay system (1).

Proof: By applying Theorem 1 and Schur complement, a sufficient condition for the closed-loop system (22) to have a specified H_∞ performance γ is that there exist symmetric positive definite matrices $P = \mathrm{diag}\{P_h, P_v\}$ and $Q = \mathrm{diag}\{Q_h, Q_v\}$ such that

$$\begin{bmatrix} -P+Q & 0 & 0 & 0 \\ 0 & -Q_h & 0 & 0 \\ 0 & 0 & -Q_v & 0 \\ 0 & 0 & 0 & -\gamma^2 I \\ A + B_2 K & A_{d_1} & A_{d_2} & B_1 \\ H + L_2 K & 0 & 0 & L_1 \end{bmatrix}$$

$$\begin{bmatrix} A^{\mathrm{T}} + K^{\mathrm{T}} B_2^{\mathrm{T}} & H^{\mathrm{T}} + K^{\mathrm{T}} L_2^{\mathrm{T}} \\ A_{d_1}^{\mathrm{T}} & 0 \\ A_{d_2}^{\mathrm{T}} & 0 \\ B_1^{\mathrm{T}} & L_1^{\mathrm{T}} \\ -P^{-1} & 0 \\ 0 & -I \end{bmatrix} < 0. \tag{25}$$

Pre- and post-multiplying both sides of the inequality (25) by $\mathrm{diag}\{P^{-1}, P^{-1}, I, I, I\}$ and denoting $W = P^{-1}$, $N = KW$ and $Y = WQW$, it follows that the inequality (25) is equal to the linear matrix inequality (23). This completes this proof. □

When time-varying norm-bounded parameter uncertainties appear in the 2-D discrete state delay system (1), that is, the system (1) becomes

$$\begin{aligned} \begin{bmatrix} x^h(i+1, j) \\ x^v(i, j+1) \end{bmatrix} &= (A + \Delta A) \begin{bmatrix} x^h(i, j) \\ x^v(i, j) \end{bmatrix} + (A_{d_1} + \Delta A_{d_1}) \\ &\quad \times x^h(i - d_1, j) + (A_{d_2} + \Delta A_{d_2}) x^v(i, j - d_2) \\ &\quad + (B_1 + \Delta B_1) w(i, j) + (B_2 + \Delta B_2) u(i, j), \\ z(i, j) &= (H + \Delta H) \begin{bmatrix} x^h(i, j) \\ x^v(i, j) \end{bmatrix} + (L_1 + \Delta L_1) w(i, j) \\ &\quad + (L_2 + \Delta L_2) u(i, j). \end{aligned} \tag{26}$$

Suppose these uncertain matrices $\Delta A, \Delta A_{d_1}, \Delta A_{d_2}$ ΔB_1, ΔB_2, ΔH, ΔL_1 and ΔL_2 be of the following form

$$\begin{aligned} [\Delta A \quad \Delta A_{d_1} \quad \Delta A_{d_2} \quad \Delta B_1 \quad \Delta B_2] &= D_1 F(i, j) [E_1 \quad E_2 \quad E_3 \quad E_4 \quad E_5], \\ [\Delta H \quad \Delta L_1 \quad \Delta L_2] &= D_2 F(i, j) [E_1 \quad E_4 \quad E_5], \end{aligned} \tag{27}$$

where D_1, D_2, E_1, E_2, E_3, E_4, and E_5 are known constant matrices that structure the uncertainties and $F(i, j) \in \mathbf{R}^{s \times t}$ is an unknown matrix function satisfying

$$F^{\mathrm{T}}(i, j) F(i, j) \le I. \tag{28}$$

We have the following robust H_∞ control results.

Theorem 3: Consider the 2-D state delay system (26) with parameter uncertainties. Given a positive scalar γ, if there exist a matrix N and symmetric positive definite matrices $W = \mathrm{diag}\{W_h, W_v\}$ and $Y = \mathrm{diag}\{Y_h, Y_v\}$, and scalar $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that

$$\begin{bmatrix} -W+Y & 0 & 0 & 0 & WA^{\mathrm{T}} + N^{\mathrm{T}} B_2^{\mathrm{T}} & WH^{\mathrm{T}} + N^{\mathrm{T}} L_2^{\mathrm{T}} & WE_1^{\mathrm{T}} + N^{\mathrm{T}} E_5^{\mathrm{T}} & WE_1^{\mathrm{T}} + N^{\mathrm{T}} E_5^{\mathrm{T}} \\ 0 & -Y_h & 0 & 0 & W_h A_{d_1}^{\mathrm{T}} & 0 & W_h E_2^{\mathrm{T}} & 0 \\ 0 & 0 & -Y_v & 0 & W_v A_{d_2}^{\mathrm{T}} & 0 & W_v E_3^{\mathrm{T}} & 0 \\ 0 & 0 & 0 & -\gamma^2 I & B_1^{\mathrm{T}} & L_1^{\mathrm{T}} & E_4^{\mathrm{T}} & E_4^{\mathrm{T}} \\ AW + B_2 N & A_{d_1} W_h & A_{d_2} W_v & B_1 & \varepsilon_1 D_1 D_1^{\mathrm{T}} - W & 0 & 0 & 0 \\ HW + L_2 N & 0 & 0 & L_1 & 0 & \varepsilon_2 D_2 D_2^{\mathrm{T}} - I & 0 & 0 \\ E_1 W + E_5 N & E_2 W_h & E_3 W_v & E_4 & 0 & 0 & -\varepsilon_1 I & 0 \\ E_1 W + E_5 N & 0 & 0 & E_4 & 0 & 0 & 0 & -\varepsilon_2 I \end{bmatrix} < 0, \tag{29}$$

then

$$u(i, j) = NW^{-1} x(i, j) \tag{30}$$

is a robust γ-suboptimal state feedback H_∞ controller for the uncertain 2-D state delay system (26).

The proof of Theorem 3 can be carried out by using Theorem 2, and hence it is omitted.

In addition, by solving the following optimization problem:

$$\begin{aligned} &\min_{W, Y, N, \varepsilon_1, \varepsilon_2} \quad \gamma^2 \\ &\quad s.\ t.\ (29), \end{aligned} \tag{31}$$

we can obtain a state feedback controller such that the H_∞ disturbance attenuation γ of the corresponding closed-loop system is minimized. This controller (30) is said to be the robust optimal H_∞ controller for the uncertain 2-D discrete state delay system (26).

4. AN ILLUSTRATIVE EXAMPLE

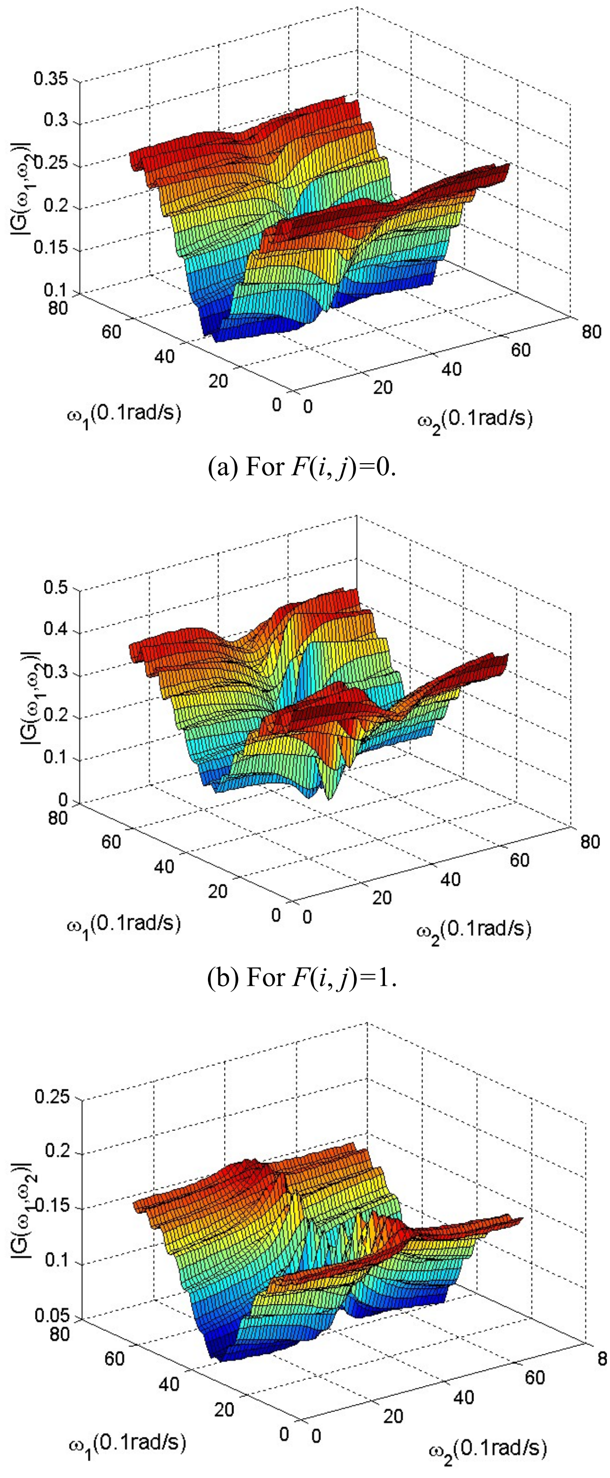
This section gives an example to illustrate the proposed results. Consider the following discrete 2-D state delay system described by (26), where

$$A = \begin{bmatrix} 0.0410 & 0.2107 \\ -0.2879 & -0.4593 \end{bmatrix}, \quad A_{d_1} = \begin{bmatrix} 0.1453 \\ 0.0824 \end{bmatrix},$$

$$A_{d_2} = \begin{bmatrix} 0.0880 \\ 0.1867 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.3092 \\ 0.2288 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.7322 \\ 0.7708 \end{bmatrix},$$

$$H=\begin{bmatrix}0.3043 & 0.0082\\ 0.0079 & 0.0950\end{bmatrix},\quad L_1=\begin{bmatrix}0.2035\\ 0.0288\end{bmatrix},$$

$$L_2=\begin{bmatrix}0.1838\\ 0.3157\end{bmatrix},\quad D_1=\begin{bmatrix}0.2\\ 0.2\end{bmatrix},\quad D_2=\begin{bmatrix}0.2\\ 0\end{bmatrix},$$



(a) For $F(i,j)$=0.

(b) For $F(i,j)$=1.

(c) For $F(i,j)$=-1.

Fig. 1. The frequency response of the disturbance transfer function.

$$E_1=\begin{bmatrix}0.2 & 0.4\end{bmatrix},\quad E_2=0.2,\quad E_3=0.2,\quad E_4=0.4,$$
$$E_5=0.4,\quad d_1=10,\quad d_2=10.$$

By applying Theorem 3 and solving the optimization problem (31), we obtain

$$W=\begin{bmatrix}2.1518 & 0\\ 0 & 3.1089\end{bmatrix},\quad Y=\begin{bmatrix}0.6157 & 0\\ 0 & 0.9183\end{bmatrix},$$
$$N=\begin{bmatrix}-0.3666 & -0.3861\end{bmatrix},$$

and γ=0.4993. Thus, the robust optimal H_∞ controller is obtained as

$$u(i,j)=\begin{bmatrix}-0.1704 & -0.1242\end{bmatrix}x(i,j). \tag{32}$$

For $F(i, j)$=0, $F(i, j)$=1 and $F(i, j)$=-1, part (a), (b) and (c) of Fig. 1 respectively show the frequency response from the disturbance input $w(i, j)$ to the controlled output $z(i, j)$ for the corresponding closed-loop system over all frequencies, i.e., $\left|G(e^{j\omega_1}, e^{j\omega_2})\right|$, $0\le\omega_1\le 2\pi$, $0\le\omega_2\le 2\pi$. The maximum value of $\left|G(e^{j\omega_1}, e^{j\omega_2})\right|$ is 0.4401 that is below the specified level of attenuation γ=0.4993.

5. CONCLUSIONS

This paper has presented an LMI approach for the H_∞ control of 2-D discrete state delay systems described by the Roesser model. The stability and H_∞ disturbance attenuation condition has been developed via the LMI approach. The design of the H_∞ controller can be recast as a convex optimization with constraints of LMI. All results can be extended to the multiple delay case.

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