Complete Identification of Isotropic Configurations of a Caster Wheeled Mobile Robot with Nonredundant/Redundant Actuation

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Abstract: In this paper, we present the complete isotropy analysis of a caster wheeled omnidirectional mobile robot (COMR) with nonredundant/redundant actuation. It is desirable for robust motion control to keep a COMR close to the isotropy but away from the singularity as much as possible. First, with the characteristic length introduced, the kinematic model of a COMR is obtained based on the orthogonal decomposition of the wheel velocities. Second, a general form of the isotropy conditions of a COMR is given in terms of physically meaningful vector quantities which specify the wheel configuration. Third, for all possible nonredundant and redundant actuation sets, the algebraic expressions of the isotropy conditions are derived so as to identify the isotropic configurations of a COMR. Fourth, the number of the isotropic configurations, the isotropic characteristic length, and the optimal initial configuration are discussed.

Keyword: Caster wheeled mobile robot, characteristic length, isotropic configuration, redundant actuation.

1. INTRODUCTION

The omnidirectional mobility of a mobile robot is required to navigate in daily life environment which is restricted in space and cluttered with obstacles. Several omnidirectional wheel mechanisms have been proposed, including universal wheels, Swedish wheels, orthogonal wheels, ball wheels, and so on [1,2]. Recently, caster wheels were employed as a practical and effective means to develop an omnidirectional mobile robot at Stanford University, which was later commercialized by Nomadic Technologies as XR4000 [3]. Since caster wheels operate without additional peripheral rollers or support structure, a caster wheeled omnidirectional mobile robot or a COMR can maintain good performance even under varying payload or ground condition.

There have been several works on the kinematic issues of a COMR. For a general form of wheeled mobile robots, a systematic kinematic modeling procedure was developed [1,2]. Regarding the minimal admissible actuation, it was shown that at least four joints out of two caster wheels should be actuated to avoid the singularity [4]. For some specific actuation sets, the global isotropic characteristics over the entire wheel configurations was considered to obtain the optimal design parameters of the mechanism [5]. For representative actuation sets, the algebraic conditions for the (local) isotropy were derived to identify the isotropic configurations [6]. On the other hand, for an omnidirectional mobile robot employing Swedish wheels, the isotropy analysis was made but the results are incomplete and need further elaboration [7].

For a COMR having $n(\geq 3)$ actuated joints, the relationship between the joint velocity, $\dot{\mathbf{\Theta}} \in \mathbf{R}^{n\times 1}$, and the task velocity, $\dot{\mathbf{x}} \in \mathbf{R}^{3\times 1}$, can be expressed in the form of $\mathbf{A}\dot{\mathbf{x}} = \mathbf{B}\dot{\mathbf{\Theta}}$, where $\mathbf{A} \in \mathbf{R}^{n\times 3}$ and $\mathbf{B} \in \mathbf{R}^{n\times n}$ are the Jacobian matrices [6]. Note that A is a function of a given wheel configuration, while $\mathbf{B}$ is always nonsingular independently of the wheel configuration. When the rank of $\mathbf{A}$ is less than three, a COMR falls into the singular configurations, in which the task velocities within the nullspace of $\mathbf{A}$ can be produced even with all the actuated joints locked [8]. On the other hand, when three singular values of $\mathbf{B}^{-1}\mathbf{A}$ become identical, a COMR reaches the isotropic configurations, in which the joint velocities required for a unity task velocity in all directions are uniform in magnitude [9]. Obviously, it is desirable for robust motion control to keep a COMR away from the singular configurations but close to the isotropic configurations, as much as possible [7].

Manuscript received March 8, 2005; revised January 11, 2006; accepted April 12, 2006. Recommended by Editorial Board member Sooyong Lee under the direction of Editor Jae-Bok Song. This work was supported by Hankuk University of Foreign Studies Research Fund of 2006.

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The purpose of this paper is to completely identify the isotropic configurations of a COMR with nonredundant/redundant actuation. This paper is organized as follows. With the characteristic length introduced [7], Section 2 presents the kinematic model based on the orthogonal decomposition of the wheel velocities. In Section 3, a general form of the isotropy conditions is given in terms of physically meaningful vector quantities specifying the wheel configuration. For all possible nonredundant and redundant actuation sets, Sections 4 and 5 derive the algebraic expressions of the isotropy conditions to identify the isotropic configurations. Section 6 discusses the number of isotropic configurations, the isotropic characteristic length, and the optimal initial configuration. Finally, the conclusion is made in Section 7.

2. KINEMATIC MODEL

Consider a COMR with three caster wheels attached to a regular triangular platform moving on the *xy*-plane, as shown in Fig. 1.

Let l be the side length of the platform with the center O_b, and three vertices, O_i, $i = 1, 2, 3$. For the i^{th} caster wheel with the center P_i, $i = 1, 2, 3$, we define the following. Let d_i and r_i be the length of the steering link and the radius of the wheel, respectively. Let θ_i and φ_i be the angles of the rotating and the steering joints, respectively. Let $\mathbf{u}_i$ and $\mathbf{v}_i$ be two orthogonal unit vectors along the steering link and the wheel axis, respectively, such that

$$\mathbf{u}_i = \begin{bmatrix} -\cos\varphi_i \\ -\sin\varphi_i \end{bmatrix}, \quad \mathbf{v}_i = \begin{bmatrix} -\sin\varphi_i \\ \cos\varphi_i \end{bmatrix}. \tag{1}$$

Note that

$$\mathbf{u}_i \mathbf{u}_i^t + \mathbf{v}_i \mathbf{v}_i^t = \mathbf{I}_2, \tag{2}$$

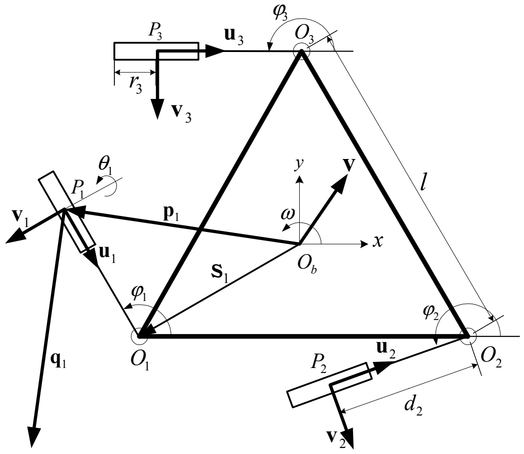


Fig. 1. A caster wheeled omnidirectional mobile robot.

$$\sum \mathbf{u}_i = \mathbf{0} \Leftrightarrow \sum \mathbf{v}_i = \mathbf{0}, \tag{3}$$

where $\mathbf{I}$ is the identity matrix and $\mathbf{0}$ is the zero vector. Let $\mathbf{p}_i$ be the vector from O_b to P_i, and $\mathbf{q}_i$ be the rotation of $\mathbf{p}_i$ by 90° counterclockwise. Note that

$$\sum \mathbf{q}_i = \mathbf{0} \Leftrightarrow \sum \mathbf{p}_i = \mathbf{0}, \tag{4}$$

$$\sum_1^3 \mathbf{p}_i = \mathbf{0} \Leftrightarrow \sum_1^3 \mathbf{u}_i = \mathbf{0}. \tag{5}$$

Let and $\mathbf{v}$ be ω the linear and the angular velocities at O_b of the platform, respectively. For the i^{th} caster wheel, $i = 1, 2, 3,$ the linear velocity at the point of contact with the ground can be expressed by

$$\mathbf{v} + \omega \mathbf{q}_i = r_i\,\dot{\theta}_i\,\mathbf{u}_i + d_i\,\dot{\varphi}_i\,\mathbf{v}_i, \quad i = 1, 2, 3. \tag{6}$$

Premultiplied by $\mathbf{u}_i^t$ and $\mathbf{v}_i^t$, we have

$$\mathbf{u}_i^t\,\mathbf{v} + \mathbf{u}_i^t\,\mathbf{q}_i\,\omega = r_i\,\dot{\theta}_i, \quad i = 1, 2, 3, \tag{7}$$

$$\mathbf{v}_i^t\,\mathbf{v} + \mathbf{v}_i^t\,\mathbf{q}_i\,\omega = d_i\,\dot{\varphi}_i, \quad i = 1, 2, 3. \tag{8}$$

Assume that $n\,(3 \le n \le 6)$ joints of a COMR are actuated. With the characteristic length, $L(>0)$, introduced [6], the kinematics of a COMR can be written as

$$\mathbf{A}\,\dot{x} = \mathbf{B}\dot{\mathbf{\Theta}}, \tag{9}$$

where $\dot{\mathbf{x}} = [\mathbf{v}\ L\ \omega]^t \in \mathbf{R}^{3\times 1}$ is the task velocity vector, and $\dot{\mathbf{\Theta}} \in \mathbf{R}^{n\times 1}$ is the joint velocity vector, and

$$\mathbf{A} = \begin{bmatrix} \mathbf{g}_1^t & \frac{1}{L}\ \mathbf{g}_1^t\ \mathbf{h}_1 \\ \vdots & \vdots \\ \mathbf{g}_n^t & \frac{1}{L}\ \mathbf{g}_n^t\ \mathbf{h}_n \end{bmatrix} \in \mathbf{R}^{n\times 3}, \tag{10}$$

$$\mathbf{B} = \begin{bmatrix} c_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & c_n \end{bmatrix} \in \mathbf{R}^{n\times n}, \tag{11}$$

are the Jacobian matrices. In (10), $\mathbf{g}_k$, $k = 1, \cdots, n,$ corresponds to either $\mathbf{u}_i$ or $\mathbf{v}_i$, $i = 1, 2, 3,$ while $\mathbf{h}_k$, $k = 1, \cdots, n,$ corresponds to $\mathbf{q}_i$, $i = 1, 2, 3.$ In (11), c_k, $k = 1, \cdots, n$, corresponds to either r_i or d_i, $i = 1, 2, 3.$ It should be mentioned that the introduction of the characteristic length L makes all three columns of A to be consistent in physical unit.

The expression of $\mathbf{g}_k^t\ \mathbf{h}_k$, $k = 1, \cdots, n,$ can be simplified as follows. In the case of the rotating joint for which $\mathbf{g}_k = \mathbf{u}_i$ and $\mathbf{h}_k = \mathbf{q}_i$, $i = 1, 2, 3$,

$$\mathbf{g}_k^{\ t}\ \mathbf{h}_k = \mathbf{u}_i^{\ t}\ \mathbf{q}_i = \mathbf{v}_i^{\ t}\ \mathbf{p}_i . \tag{12}$$

And, in the case of the steering joint for which $\mathbf{g}_k = \mathbf{v}_i$ and $\mathbf{h}_k = \mathbf{q}_i$, $i = 1, 2, 3$,

$$\mathbf{g}_k^{\ t}\ \mathbf{h}_k = \mathbf{v}_i^{\ t}\ \mathbf{q}_i = -\mathbf{u}_i^{\ t}\ \mathbf{p}_i . \tag{13}$$

It is worthwhile to mention that our kinematic modeling of a COMR does not involve matrix inversion, unlike the transfer method proposed in [2,5]. For a given task velocity, the instantaneous motion of the wheel is decomposed into two orthogonal components: the instantaneous motion of the rotating joint and the instantaneous motion of the steering joint. The resulting kinematic model allows us to perform a geometric and intuitive analysis on the isotropy of a COMR.

3. ISOTROPY CONDITIONS

Based on (9), the necessary and sufficient condition for the isotropy of a COMR is given by

$$(\mathbf{B}^{-1}\ \mathbf{A})^{\ t}\ \ (\mathbf{B}^{-1}\ \mathbf{A}) \propto \mathbf{I}_3 . \tag{14}$$

In [5], it is found to be optimal for global isotropic characteristics that three caster wheels are identical to have the steering link length equal to the wheel radius, that is,

$$c_k = d > 0, \quad k = 1, \cdots, n \Leftrightarrow \mathbf{B} \propto \mathbf{I}_6 . \tag{15}$$

Under the assumption of (15), (14) becomes

$$\mathbf{A}^t\ \mathbf{A} \propto \mathbf{I}_3 . \tag{16}$$

Note that in fact (15) and (16) are the sufficient conditions for the isotropy of a COMR.

From (10) and (16), the isotropy condition on $\mathbf{A}$ is obtained by

$$\mathbf{A}^t\ \mathbf{A} = \frac{n}{2}\ \mathbf{I}_3 , \tag{17}$$

which leads to the following three conditions:

$$\begin{aligned} &\mathbf{C1} : \sum_1^n \mathbf{g}_k\ \mathbf{g}_k^{\ t} = \frac{n}{2}\mathbf{I}_2 \in \mathbf{R}^{2\times 2}, \\ &\mathbf{C2} : \sum_1^n (\mathbf{g}_k^{\ t}\ \mathbf{h}_k)\mathbf{g}_k = \mathbf{0} \in \mathbf{R}^{2\times 1}, \\ &\mathbf{C3} : \frac{1}{L^2}\sum_1^n (\mathbf{g}_k^{\ t}\ \mathbf{h}_k)^{\ 2} = \frac{n}{2} \in \mathbf{R}^{1\times 1}. \end{aligned} \tag{18}$$

In general, **C1** and **C2** correspond to three and two scalar constraints, respectively, which are imposed on three steering joint angles, φ_k, $k = 1, 2, 3$. Thus, the isotropy of a COMR can occur only at specific values of φ_k, $k = 1, 2, 3$, called isotropic configurations, for which **C1** and **C2** are satisfied simultaneously. For a given isotropic configuration, **C3**, corresponding to one scalar constraint, determines the characteristic length required for the isotropy, denoted by L_{iso}.

In what follows, it is assumed that a COMR has three identical caster wheels having the steering link length equal to the wheel radius.

4. ISOTROPY ANALYSIS FOR NONREDUNDANT ACTUATION

4.1. Nonredundant actuation sets

A COMR with nonredundant acuation can have three actuated joints $(n = 3)$, each of which can be either rotating or steering one. According to the number of active wheels and the combination of actuated joints, all possible nonredundant actuation sets, Θ, can be divided into three groups, denoted by NAG I, II, and III, as listed in Table 1.

4.2. Isotropy analysis for NAG I

Consider $\mathbf{\Theta} = \{\theta_1, \theta_2, \theta_3\}$ where three rotating joints of three caster wheels are actuated, for which $[\mathbf{g}_1\ \mathbf{g}_2\ \mathbf{g}_3] = [\mathbf{u}_1\ \mathbf{u}_2\ \mathbf{u}_3]$.

Under the condition of **C1**, we have

$$\sum_{k=1}^{3} \mathbf{u}_k \mathbf{u}_k^{\ t} = 1.5\ \ \mathbf{I}_2 , \tag{19}$$

which is

$$\begin{aligned} c_1^{\ 2} + c_1^{\ 2} + c_1^{\ 2} &= 1.5, \\ c_1 s_1 + c_1 s_1 + c_1 s_1 &= 0.0, \end{aligned} \tag{20}$$

Table 1. Three nonredundant actuation groups.

Number of actuated joints	Number of active wheels	Actuation set	Nonredundant actuation group
$n = 3$	3	$\mathbf{\Theta} = \{\theta_1, \theta_2, \theta_3\}$	NAG I
		$\mathbf{\Theta} = \{\varphi_1, \varphi_2, \varphi_3\}$	
		$\mathbf{\Theta} = \{\varphi_1, \theta_2, \theta_3\}$	NAG II
		$\mathbf{\Theta} = \{\varphi_1, \varphi_2, \theta_3\}$	
	2	$\mathbf{\Theta} = \{\theta_1, \varphi_1, \theta_2\}$	NAG III
		$\mathbf{\Theta} = \{\theta_1, \varphi_1, \varphi_2\}$	

where $c_k = \cos(\varphi_k)$ and $s_k = \sin(\varphi_k)$, $k = 1, 2, 3$.

There are eight different distributions of $\{u_k, k = 1, 2, 3\}$ satisfying (20), which can be divided into two distinctive groups characterized, respectively, by

$$\varphi_2 = \varphi_1 + \frac{2}{3}\pi,\ \varphi_1 - \frac{\pi}{3},\ \varphi_3 = \varphi_1 + \frac{\pi}{3},\ \varphi_1 - \frac{2}{3}\pi, \quad (21)$$

$$\varphi_2 = \varphi_1 + \frac{\pi}{3},\ \varphi_1 - \frac{2}{3}\pi,\ \varphi_3 = \varphi_1 + \frac{2}{3}\pi,\ \varphi_1 - \frac{\pi}{3}, \quad (22)$$

as shown in Fig. 2. The first group of four distributions, characterized by (21), is common in that u_1, u_2, and u_3 lie on three sides of a regular triangle in counterclockwise order, as shown in Fig. 2(a). On the other hand, the second group of four distributions, characterized by (22), is common in that u_1, u_2, and u_3 lie on three sides of a regular triangle in clockwise order, as shown in Fig. 2(b).

Under the condition of , we have

$$\sum_1^3 \left(\mathbf{u}_k^t \ \ \mathbf{q}_k\right)\mathbf{u}_k = \sum_1^3 \left(\mathbf{v}_k^t \ \ \mathbf{p}_k\right)\mathbf{u}_k = \mathbf{0}, \quad (23)$$

which is equivalent to

$$\sum_1^3 \left(\mathbf{v}_k^t \ \ \mathbf{q}_k\right)\mathbf{v}_k = \sum_1^3 a_k \ \mathbf{v}_k = \mathbf{0}, \quad (24)$$

where $\alpha_k = \mathbf{v}_k^t \ \mathbf{p}_k$, $k = 1, 2, 3$, is the projection of $\mathbf{p}_k$ onto $\mathbf{v}_k$. For the first group of four distributions characterized by (21), it can be shown that [5]

$$|\ \alpha_1\ | = |\ \alpha_2\ | = |\ \alpha_3\ | = \alpha, \quad (25)$$

$$\alpha_1\ \mathbf{v}_1 + \alpha_2\ \mathbf{v}_2 + \alpha_3\ \mathbf{v}_3 = \mathbf{0}. \quad (26)$$

While **C1** places two scalar constraints, given by (20), on three variables, φ_1, φ_2, and φ_3, **C2** does not place additional constraint. As a result, there are in-

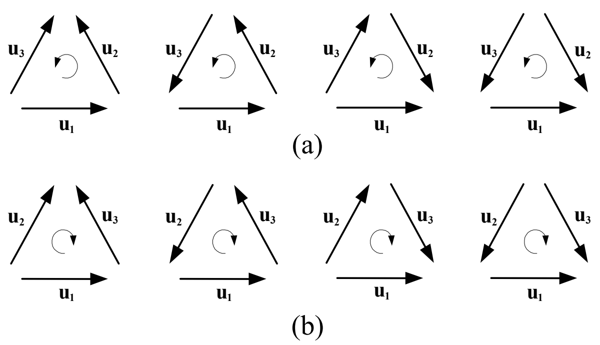
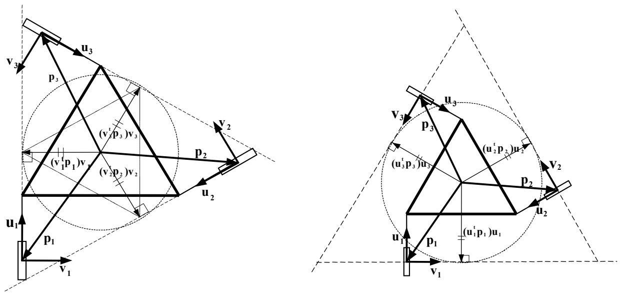


Fig. 2. Two distinctive groups of the distributions of $\{\mathbf{u}_k,\ k = 1, 2, 3\}$: (a) counterclockwise order and (b) clockwise order.



(a) (b)

Fig. 3. Isotropic configurations for NAG I : (a) $\mathbf{\Theta} = \{\theta_1, \theta_2, \theta_3\}$ and (b) $\mathbf{\Theta} = \{\varphi_1, \varphi_2, \varphi_3\}$.

finitely many isotropic configurations in general. Fig. 3(a) illustrates an isotropic configuration of a COMR, where three steering links form a regular triangle centered at the platform center, inscribed by a circle of radius α.

On the other hand, it can be shown that the second group of four distributions characterized by (22) cannot satisfy (24), so that the isotropy of **A** cannot be achieved.

Similar analysis to the above can be made for $\mathbf{\Theta} = \{\varphi_1, \varphi_2, \varphi_3\}$, where three steering joints of three caster wheels are actuated. Fig. 3(b) illustrates an isotropic configuration of a COMR, where three wheel axes form a regular triangle centered at the platform center, which is inscribed by a circle of radius $\beta (= |\mathbf{u}_1^t\ \mathbf{p}_1| = |\mathbf{u}_2^t\ \mathbf{p}_2| = |\mathbf{u}_3^t\ \mathbf{p}_3|)$.

4.3. Isotropy analysis for NAG II

Consider $\mathbf{\Theta} = \{\varphi_1, \theta_2, \theta_3\}$ where one steering and two rotating joints of three caster wheels are actuated, for which $[\mathbf{g}_1\ \mathbf{g}_2\ \mathbf{g}_3] = [\mathbf{v}_1\ \mathbf{u}_2\ \mathbf{u}_3]$.

First, under **C1**, we have

$$\mathbf{v}_1\mathbf{v}_1^t + \mathbf{u}_2\mathbf{u}_2^t + \mathbf{u}_3\mathbf{u}_3^t = 1.5\mathbf{I}_2. \quad (27)$$

Next, under **C2**, we have

$$(\mathbf{v}_1^t\mathbf{q}_1)\mathbf{v}_1 + (\mathbf{u}_2^t\mathbf{q}_2)\mathbf{u}_2 + (\mathbf{u}_3^t\mathbf{q}_3)\mathbf{u}_3 = \mathbf{0}, \quad (28)$$

or

$$(\mathbf{u}_1^t\mathbf{p}_1)\mathbf{u}_1 + (\mathbf{v}_2^t\mathbf{p}_2)\mathbf{v}_2 + (\mathbf{v}_3^t\mathbf{p}_3)\mathbf{v}_3 = \mathbf{0}. \quad (29)$$

With (27) being held, it can be shown that (29) cannot be satisfied unless d is equal to zero [5]. This tells that the isotropy of **A** can be achieved only when caster wheels reduce to conventional wheels without steering link. Fig. 4(a) illustrates an isotropic configuration of a conventional wheeled mobile robot.

Similar analysis to the above can be made for $\mathbf{\Theta} = \{\varphi_1, \varphi_2, \theta_3\}$, where two steering and one rotating

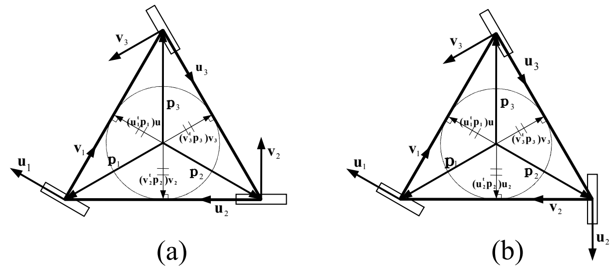


Fig. 4. With $d=0$, isotropic configurations for NAG II: (a) $\mathbf{\Theta}=\{\varphi_1,\ \theta_2,\ \theta_3\}$ and (b) $\mathbf{\Theta}=\{\varphi_1,\ \varphi_2,\ \theta_3\}$.

joints of three caster wheels are actuated. Fig. 4(b) illustrates an isotropic configuration of a conventional wheeled mobile robot.

4.4. Isotropy analysis for NAG III

Consider $\mathbf{\Theta}=\{\theta_1,\ \varphi_1,\ \theta_2\}$ where both rotating and steering joints of one caster wheel and the rotating joint of another caster wheel are actuated, for which $[\mathbf{g}_1\ \mathbf{g}_2\ \mathbf{g}_3]=[\mathbf{u}_1\ \mathbf{v}_1\ \mathbf{u}_2]$.

First, under **C1**, we have

$$\mathbf{u}_1\mathbf{u}_1^t+\mathbf{v}_1\mathbf{v}_1^t+\mathbf{u}_2\mathbf{u}_2^t=1.5\mathbf{I}_2, \quad (30)$$

which is

$$c_2^2=0.5,\quad c_2s_2=0.0. \quad (31)$$

There does not exist φ_2 satisfying (31), and the isotropy of **A** cannot be achieved at all.

Similar analysis to the above can be made for $\mathbf{\Theta}=\{\theta_1,\ \varphi_1,\ \varphi_2\}$, where both rotating and steering joints of one caster wheel and the steering joint of another caster wheel are actuated.

5. ISOTROPY ANALYSIS FOR REDUNDANT ACTUATION

5.1. Redundant actuation sets

A COMR with redundant acuation can have four, five and six actuated joints $(n=4,\ 5,\ 6)$, each of which can be either rotating or steering one. According to the number and combination of actuated joints and the number of active wheels, all possible redundant actuation sets, Θ, can be divided into five groups, denoted by RAG I, II, III, IV, and V, as listed in Table 2.

5.2. Isotropy analysis for RAG I

Consider $\mathbf{\Theta}=\{\theta_1,\ \varphi_1,\ \theta_2,\ \varphi_2\}$ where both rotating and steering joints of two caster wheels are actuated,

Table 2. Five redundant actuation groups.

Number of actuated joints	Number of active wheels	Actuation set	Redundant actuation group
$n=4$	2	$\mathbf{\Theta}=\{\theta_1,\ \varphi_1,\ \theta_2,\ \varphi_2\}$	RAG I
	3	$\mathbf{\Theta}=\{\theta_1,\ \varphi_1,\ \theta_2,\ \theta_3\}$	RAG II
		$\mathbf{\Theta}=\{\theta_1,\ \varphi_1,\ \varphi_2,\ \varphi_3\}$	
		$\mathbf{\Theta}=\{\theta_1,\ \varphi_1,\ \theta_2,\ \varphi_3\}$	RAG III
$n=5$	3	$\mathbf{\Theta}=\{\theta_1,\ \varphi_1,\ \theta_2,\ \varphi_2,\ \theta_3\}$	RAG IV
		$\mathbf{\Theta}=\{\theta_1,\ \varphi_1,\ \theta_2,\ \varphi_2,\ \varphi_3\}$	
$n=6$	3	$\mathbf{\Theta}=\{\theta_1,\ \varphi_1,\ \theta_2,\ \varphi_2,\ \theta_2,\ \varphi_3\}$	RAG V

for which $[\mathbf{g}_1\ \mathbf{g}_2\ \mathbf{g}_3\ \mathbf{g}_4]=[\mathbf{u}_1\ \mathbf{v}_1\ \mathbf{u}_2\ \mathbf{v}_2]$.

First, under **C1**, we have

$$\sum_1^2(\mathbf{u}_k\mathbf{u}_k^t+\mathbf{u}_k\mathbf{u}_k^t)=2\mathbf{I}_2, \quad (32)$$

which always holds. Next, under **C2**, we have

$$\sum_1^2\{(\mathbf{u}_k^t\mathbf{q}_k)\mathbf{u}_k+(\mathbf{v}_k^t\mathbf{q}_k)\mathbf{v}_k\}=\sum_1^2\mathbf{q}_k=0, \quad (33)$$

or

$$\mathbf{p}_1+\mathbf{p}_2=\mathbf{0}, \quad (34)$$

which yields

$$\varphi_1=\arcsin\left(\frac{1}{2\sqrt{3}}\ \frac{l}{d}\right),\quad \varphi_2=\pi-\varphi_1. \quad (35)$$

Since **C1** places no constraint and **C2** places two scalar constraints, given by (34), on two variables, φ_1 and φ_2, in general, there are multiple isotropic configurations independently of φ_3. Fig. 5 illustrates an isotropic configuration, where the steering links of two caster wheels are symmetric with respect to y-axis,

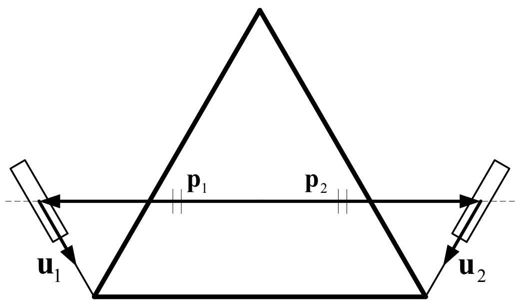


Fig. 5. An isotropic configurations for $\mathbf{\Theta}=\{\theta_1,\ \varphi_1,\ \theta_2,\ \varphi_2\}$ belonging to RAG I.

with the centers of two caster wheels and the center of the platform lying on the line of $y = \frac{l}{2\sqrt{3}}$. Note that the isotropic configuration does not exist if the steering link length is less than $\frac{l}{2\sqrt{3}}$.

5.3. Isotropy analysis for RAG II

Consider $\mathbf{\Theta} = \{\theta_1, \varphi_1, \theta_2, \theta_3\}$ where both rotating and steering joints of one caster wheel and two rotating joints of two caster wheels are actuated, for which $[\mathbf{g}_1\ \mathbf{g}_2\ \mathbf{g}_3\ \mathbf{g}_4] = [\mathbf{u}_1\ \mathbf{v}_1\ \mathbf{u}_2\ \mathbf{u}_3]$.

First, under **C1**, we have

$$\mathbf{u}_2\mathbf{u}_2^t + \mathbf{u}_3\mathbf{u}_3^t = \mathbf{I}_2, \tag{36}$$

which is

$$c_2 s_2 + c_3 s_3 = 0.0, \tag{37}$$

hence

$$\varphi_3 = \varphi_2 \pm \frac{\pi}{2}. \tag{38}$$

Note that $\mathbf{u}_2$ and $\mathbf{u}_3$ are perpendicular to each other, and so are $\mathbf{v}_2$ and $\mathbf{v}_3$. Next, under **C2**, we have

$$\mathbf{q}_1 + (\mathbf{u}_2^t\,\mathbf{q}_2)\mathbf{u}_2 + (\mathbf{u}_3^t\,\mathbf{q}_3)\mathbf{u}_3 = \mathbf{0}, \tag{39}$$

or

$$\mathbf{p}_1 + (\mathbf{v}_2^t\,\mathbf{p}_2)\mathbf{v}_2 + (\mathbf{v}_3^t\,\mathbf{p}_3)\mathbf{v}_3 = \mathbf{0}. \tag{40}$$

Since **C1** places one scalar constraint, given by (37), and **C2** places two scalar constraints, given by (40), on three variables φ_1, φ_2, and φ_3, there are multiple isotropic configurations in general. Fig. 6(a) illustrates an isotropic configuration, where the steering links of two caster wheels with actuated rotating joint are perpendicular to each other and the center of the other caster wheel with actuated rotating and steering joints is located under the constraint of (40).

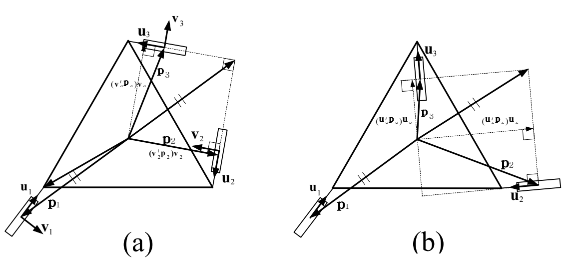

(a) (b)

Fig. 6. Isotropic configurations for RAG II: (a) $\mathbf{\Theta} = \{\theta_1, \varphi_1, \theta_2, \theta_3\}$ and (b) $\mathbf{\Theta} = \{\theta_1, \varphi_1, \varphi_2, \varphi_3\}$.

Similar analysis to the above can be made for $\mathbf{\Theta} = \{\theta_1, \varphi_1, \varphi_2, \varphi_3\}$ where both rotating and steering joints of one caster wheel and two steering joints of two caster wheels are actuated. Fig. 6(b) illustrates an isotropic configuration, where the steering links of two caster wheels with actuated steering joint are perpendicular to each other, and the center of the other caster wheel with actuated rotating and steering joints is located under the following constraint:

$$\mathbf{p}_1 + (\mathbf{u}_2^t\,\mathbf{p}_2)\mathbf{u}_2 + (\mathbf{u}_3^t\,\mathbf{p}_3)\mathbf{u}_3 = \mathbf{0}. \tag{41}$$

5.4. Isotropy analysis for RAG III

Consider $\mathbf{\Theta} = \{\theta_1, \varphi_1, \theta_2, \varphi_3\}$ where both rotating and steering joints of one caster wheel and one rotating and one steering joints of two caster wheels are actuated, for which $[\mathbf{g}_1\,\mathbf{g}_2\,\mathbf{g}_3\,\mathbf{g}_4] = [\mathbf{u}_1\,\mathbf{v}_1\,\mathbf{u}_2\,\mathbf{v}_3]$.

First, under **C1**, we have

$$\mathbf{u}_2\,\mathbf{u}_2^t + \mathbf{v}_3\,\mathbf{v}_3^t = \mathbf{I}_2, \tag{42}$$

which is

$$c_2 s_2 - c_3 s_3 = 0.0, \tag{43}$$

hence

$$\varphi_3 = \varphi_2. \tag{44}$$

Next, under **C2**, we have

$$\mathbf{q}_1 + (\mathbf{u}_2^t\,\mathbf{q}_2)\mathbf{u}_2 + (\mathbf{v}_3^t\,\mathbf{q}_3)\mathbf{v}_3 = \mathbf{0}, \tag{45}$$

or

$$\mathbf{q}_1 + (\mathbf{u}_2^t\,\mathbf{q}_2)\mathbf{u}_2 + (\mathbf{v}_3^t\,\mathbf{q}_3)\mathbf{v}_3 = \mathbf{0}. \tag{46}$$

Since **C1** places one scalar constraint, given by (43), and **C2** places two scalar constraints, given by (46), on three variables, φ_1, φ_2, and φ_3, there are multiple isotropic configurations in general. Fig. 7 illustrates an isotropic configuration, where the steering links of one caster wheel with actuated

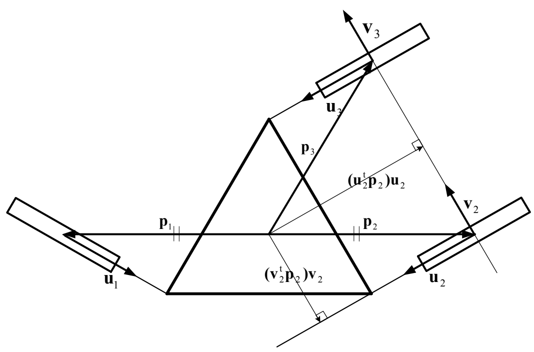


Fig. 7. An isotropic configuration for $\mathbf{\Theta} = \{\theta_1, \varphi_1, \theta_2, \varphi_3\}$ belonging to RAG IV.

rotating joint and another caster wheel with actuated steering joint are parallel to each other, and the center of the other caster wheel with actuated rotating and steering joints is located under the constraint of (46).

5.5. Isotropy analysis for RAG IV

Consider $\mathbf{\Theta}=\{\theta_1, \varphi_1, \theta_2, \varphi_2, \theta_3\}$ where both rotating and steering joints of two caster wheels and the rotating joint of one caster wheel are actuated, for which $[\mathbf{g}_1\ \mathbf{g}_2\ \mathbf{g}_3\ \mathbf{g}_4\ \mathbf{g}_5]=[\mathbf{u}_1\ \mathbf{v}_1\ \mathbf{u}_2\ \mathbf{v}_2\ \mathbf{u}_3]$.

First, under **C1**, we have

$$\mathbf{u}_3\mathbf{u}_3{}^t = 0.5\mathbf{I}_2\,, \tag{47}$$

which is

$$c_3{}^2 = 0.5, \quad c_3 s_3 = 0.0\,. \tag{48}$$

There does not exist φ_3 satisfying (48) and the isotropy of **A** cannot be achieved at all.

Similar analysis to the above can be made for $\mathbf{\Theta}=\{\theta_1, \varphi_1, \theta_2, \varphi_2, \varphi_3\}$ where both rotating and steering joints of two caster wheels and the steering joint of one caster wheel are actuated.

5.6. Isotropy analysis for RAG V

Consider $\mathbf{\Theta}=\{\theta_1, \varphi_1, \theta_2, \varphi_2, \theta_3, \varphi_3\}$ where both rotating and steering joints of three caster wheels are fully actuated, for which $[\mathbf{g}_1\ \mathbf{g}_2\ \mathbf{g}_3\ \mathbf{g}_4\ \mathbf{g}_5\ \mathbf{g}_6]=[\mathbf{u}_1\ \mathbf{v}_1\ \mathbf{u}_2\ \mathbf{v}_2\ \mathbf{u}_3\ \mathbf{v}_3]$.

First, **C1** holds always. Next, under **C2**, we have

$$\sum_1^3 \mathbf{p}_k = \mathbf{0}\,, \tag{49}$$

which yields

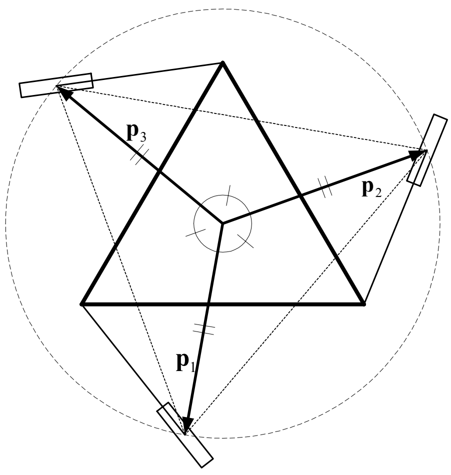


Fig. 8. An isotropic configuration for $\mathbf{\Theta}=\{\theta_1, \varphi_1, \theta_2, \varphi_2, \theta_3, \varphi_3\}$ belonging to RAG V.

$$\varphi_2 = \varphi_1 \pm \frac{2\pi}{3}, \quad \varphi_3 = \varphi_1 \mp \frac{2\pi}{3}\,. \tag{50}$$

Since **C1** places no constraints and **C2** places two scalar constraints, given by (49), on three variables, φ_1, φ_2, and φ_3, there are infinitely many isotropic configurations in general. Fig. 8 illustrates an isotropic configuration of a COMR. where the centers of three caster wheels are symmetric with respect to the center of the platform.

6. SOME DISCUSSIONS

6.1. Number of isotropic configurations

Depending on the selection of actuated joints, the number of isotropic configurations which satisfy **C1** and **C2** can be either none, multiple(finite), or infinite. Table 3 lists the nonredundant and the redundant actuation sets resulting in more than one isotropic configuration. From Table 3, the following observations can be made. When the actuation of three caster wheels are homogeneous, including $\mathbf{\Theta}=\{\theta_1, \theta_2, \theta_3\}$, $\{\varphi_1, \varphi_2, \varphi_3\}$, and $\{\theta_1, \varphi_1, \theta_2, \varphi_2, \theta_3, \varphi_3\}$, there are infinitely many isotropic configuretions. When the number of actuated joints are redundant, including $\mathbf{\Theta}=\{\theta_1, \varphi_1, \theta_2, \theta_3\}$, $\{\theta_1, \varphi_1, \varphi_2, \varphi_3\}$, $\{\theta_1, \varphi_1, \theta_2, \varphi_3\}$, and $\{\theta_1, \varphi_1, \theta_2, \varphi_2\}$, there are multiple isotropic configurations. The only two exceptions are $\mathbf{\Theta}=\{\theta_1, \varphi_1, \theta_2, \varphi_2, \theta_3\}$ and $\{\theta_1, \varphi_1, \theta_2, \varphi_2, \varphi_3\}$. It should be mentioned that both homogeneity in wheel actuation and redundancy in joint actuation play a significant role for enhancing the isotropy of a COMR.

6.2. Isotropic characteristic length

As described in Section 3, the isotropy of a COMR can be achieved under three conditions, **C1**, **C2**, and **C3**. Once an isotropic configuration is identified under **C1** and **C2**, the characteristic length required for the isotropy, L_{iso}, can be determined under **C3**.

As an example, let us consider the case of $\mathbf{\Theta}=\{\theta_1, \theta_2, \theta_3\}$. Under **C3**, we have

$$\frac{1}{L^2}\sum_1^3 (\mathbf{u}_k{}^t\ \mathbf{q}_k)^2 = \frac{1}{L^2}\sum_1^3 (\mathbf{v}_k{}^t\ \mathbf{p}_k)^2 = 1.5\,. \tag{51}$$

With (21) being held, from (51), the characteristic length of an isotropic COMR is obtained by

$$L_{iso} = (|\,\mathbf{v}_1{}^t\mathbf{p}_1\,| = |\,\mathbf{v}_2{}^t\ \mathbf{p}_2\,| = |\,\mathbf{v}_3{}^t\mathbf{p}_3\,|)\,. \tag{52}$$

For all actuation sets with more than one isotropic configuration, Table 3 also lists the resulting isotropic characteristic length L_{iso}. Note that the isotropy of a COMR cannot be achieved unless $L = L_{iso}$.

Table 3. All actuation sets resulting in more than one isotropic configuration.

Actuation group	Actuation set $\mathbf{\Theta}$	Number of isotropic configurations	Isotropic characteristic length L_{iso}
NAG I	$\{\theta_1, \theta_2, \theta_3\}$	Infinite	$(\lvert \mathbf{v}_1{}^t \mathbf{p}_1 \rvert = \lvert \mathbf{v}_2{}^t \mathbf{p}_2 \rvert = \lvert \mathbf{v}_3{}^t \mathbf{p}_3 \rvert)$
	$\{\varphi_1, \varphi_2, \varphi_3\}$	Infinite	$(\lvert \mathbf{u}_1{}^t \mathbf{p}_1 \rvert = \lvert \mathbf{u}_2{}^t \mathbf{p}_2 \rvert = \lvert \mathbf{u}_3{}^t \mathbf{p}_3 \rvert)$
RAG I	$\{\theta_1, \varphi_1, \theta_2, \varphi_2\}$	Multiple	$(\lVert \mathbf{p}_1 \rVert = \lVert \mathbf{p}_2 \rVert)$
RAG II	$\{\theta_1, \varphi_1, \theta_2, \theta_3\}$	Multiple	$\left(\lVert \mathbf{p}_1 \rVert = \sqrt{\left(\mathbf{v}_2{}^t \mathbf{p}_2\right)^2 + \left(\mathbf{v}_3{}^t \mathbf{p}_3\right)^2}\right)$
	$\{\theta_1, \varphi_1, \varphi_2, \varphi_3\}$	Multiple	$\left(\lVert \mathbf{p}_1 \rVert = \sqrt{\left(\mathbf{u}_2{}^t \mathbf{p}_2\right)^2 + \left(\mathbf{u}_3{}^t \mathbf{p}_3\right)^2}\right)$
RAG III	$\{\theta_1, \varphi_1, \theta, \varphi_3\}$	Multiple	$\left(\lVert \mathbf{p}_1 \rVert = \sqrt{\left(\mathbf{v}_2{}^t \mathbf{p}_2\right)^2 + \left(\mathbf{u}_3{}^t \mathbf{p}_3\right)^2}\right)$
RAG V	$\{\theta_1, \varphi_1, \theta_2, \varphi_2, \theta_3, \varphi_3\}$	Infinite	$(\lVert \mathbf{p}_1 \rVert = \lVert \mathbf{p}_2 \rVert = \lVert \mathbf{p}_3 \rVert)$

6.3. Optimal initial configuration

For a given task velocity trajectory, the wheel configurations are subject to undergo different changes depending on the initial configuration chosen. Unless the initial configuration is set carefully, a COMR may suffer from poor isotropic characteristics during task execution, which is undesirable for robust motion control. To find the optimal initial configuration with minimal computation, a simple but effective measure should be devised, which can evaluate the isotropic characteristics of a given wheel configuration. As an example, let us consider the case of $\mathbf{\Theta} = \{\theta_1, \varphi_1, \theta_2, \varphi_2, \theta_3, \varphi_3\}$.

Suppose that a task velocity trajectory over some time interval, $\dot{\mathbf{x}}(t),\ t \in [0\ \ T]$, with the prespecified initial position, $\mathbf{x}(0)$, is given to a COMR. Using (8), the steering joint angle at time t, $\varphi_i(t)$, of the i^{th} caster wheel is obtained by

$$\varphi_i(t) = \varphi_i(0) + \int_0^t \mathbf{a}_i{}^t(\varphi_i(\tau))\ \dot{\mathbf{x}}(\tau) d\tau,\ i = 1, 2, 3, \quad (53)$$

where $\mathbf{a}_i{}^t(\varphi_i) = [\mathbf{v}_i{}^t(\varphi_i)\ \ \mathbf{v}_k{}^t(\varphi_i)\ \ \mathbf{q}_k(\varphi_i)]$. Note that $\varphi_i(t)$, $i = 1, 2, 3$, is sensitive to its initial condition $\varphi_i(0)$. The configuration trajectory during task execution, Π, can be described as

$$\begin{aligned} \Pi &= \{(\varphi_1(t),\ \varphi_2(t),\ \varphi_3(t)),\ t \in [0\ \ T]\} \\ &= \Pi(\varphi_1(0),\ \varphi_2(0),\ \varphi_3(0)). \end{aligned} \quad (54)$$

Note that Π is a function of the initial configuration, $(\varphi_1(0),\ \varphi_2(0),\ \varphi_3(0))$.

With the prior knowledge of the isotropic configurations, we propose to evaluate the isotropic characteristics of a given wheel configuration based on the distance from the isotropic configurations. With $\hat{\varphi}_i = \varphi_i - \varphi_1$, $i = 2, 3$, the isotropic configurations, given by (50), can be expressed as two points in $(\hat{\varphi}_2,\ \hat{\varphi}_3)$-space, $(\frac{2\pi}{3}, -\frac{2\pi}{3})$ and $(-\frac{2\pi}{3}, \frac{2\pi}{3})$. For a given wheel configuration at time t, $(\varphi_1(t),\ \varphi_2(t),\ \varphi_3(t))$, the local isotropy measure, denoted by $\mathrm{dist}(t)$, can be defined as the smaller value of the weighted distances from $(\hat{\varphi}_2(t),\ \hat{\varphi}_3(t))$ to the two isotropic points. A proper weighting would be a function which returns zero when $(\hat{\varphi}_2(t),\ \hat{\varphi}_3(t)) = (\pm\frac{2\pi}{3},\ \mp\frac{2\pi}{3})$, and returns larger value as $\hat{\varphi}_2(t)$ or $\hat{\varphi}_3(t)$ approaches to 0 or $\pm\pi$, or $(\hat{\varphi}_2(t),\ \hat{\varphi}_3(t)) = (\pm\frac{2\pi}{3},\ \pm\frac{2\pi}{3})$.

Now, for a given configuration trajectory, Π $(\varphi_1(0),\ \varphi_2(0),\ \varphi_3(0))$, the global isotropy measure, denoted by $\mathrm{DIST}(\Pi)$, can be defined as

$$\mathrm{DIST}(\Pi) = \max_{t \in [0\ \ T]} \mathrm{dist}(t). \quad (55)$$

Geometrically, $\mathrm{DIST}(\Pi)$ can be interpreted as the radius of the smallest circle in $(\hat{\varphi}_2,\ \hat{\varphi}_3)$-space, which contains all the deviations from the isotropic configurations along Π. Finally, among all possible configuration trajectories resulting from different initial configurations, the optimal initial configuration, $(\varphi_1{}^{opt}(0),\ \varphi_2{}^{opt}(0),\ \varphi_3{}^{opt}(0))$, can be determined through the optimization given below:

$$\begin{aligned} &\min_{\forall(\varphi_1(0),\ \varphi_2(0),\ \varphi_3(0))} \mathrm{DIST}(\Pi) \\ &= \min_{\forall(\varphi_1(0),\ \varphi_2(0),\ \varphi_3(0))} [\max_{t \in [0\ \ T]} \mathrm{dist}(t)]. \end{aligned} \quad (56)$$

Note that for a given Π, computing dist(t) can be stopped immediately after the value of dist(t) becomes greater than or equal to the smallest among the global isotropy measures already computed. By choosing $(\varphi_1^{opt}(0), \varphi_2^{opt}(0), \varphi_3^{opt}(0))$, the maximal deviation from the isotropic configurations during task execution can be kept small as much as possible, which is desirable for robust motion control.

Since the initial configuration optimization for improved isotropic characteristics requires nested multiple loops, the associated computational cost depends heavily on how to devise a local isotropy measure within the innermost loop. The local isotropy measure, dist(t), which is devised using the prior knowledge of the isotropic configurations, is not only physically meaningful but also simple in computation. An alternative measure without such knowledge would be the condition number of the Jacobian matrix, **A**, which requires far more expensive computation than dist(t).

7. CONCLUSION

This paper presented the complete isotropy analysis of a caster wheeled omnidirectional mobile robot (COMR) with nonredundant and redundant actuation. All possible actuation sets with different number and combination of rotating and steering joints were considered. First, with the characteristic length introduced, the kinematic model was obtained. Second, a general form of the isotropy conditions was given in terms of physically meaningful vector quantities. Third, for all possible nonredundant and redundant actuation sets, the algebraic expressions of the isotropy conditions were derived so as to identify the isotropic configurations completely. Fourth, the number of the isotropic configurations, the isotropic characteristic length, and the optimal initial configuration were discussed. We hoped that the isotropy analysis made in this paper can serve for better design and control of a COMR with improved isotropic characteristics.

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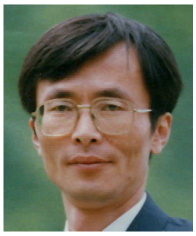

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