Robust Stabilization of Uncertain Nonlinear Systems via Fuzzy Modeling and Numerical Optimization Programming

Jongbae Lee, Chang-Woo Park*, Ha-Gyeong Sung, and Joonhong Lim

Abstract: This paper presents the robust stability analysis and design methodology of the fuzzy feedback linearization control systems. Uncertainty and disturbances with known bounds are assumed to be included in the Takagi-Sugeno (TS) fuzzy models representing the nonlinear plants. L_2 robust stability of the closed system is analyzed by casting the systems into the diagonal norm bounded linear differential inclusions (DNLDI) formulation. Based on the linear matrix inequality (LMI) optimization programming, a numerical method for finding the maximum stable ranges of the fuzzy feedback linearization control gains is also proposed. To verify the effectiveness of the proposed scheme, the robust stability analysis and control design examples are given.

Keywords: L_2 robust stability, feedback linearization, fuzzy control, linear matrix inequalities, Takagi-Sugeno fuzzy model.

1. INTRODUCTION

A fuzzy model has excellent capability in a nonlinear system description and is particularly suitable for the complex or uncertain system [1]. By using this property of the fuzzy models, the research on the fuzzy feedback linearization scheme has been conducted because the nonlinearity can be efficiently modeled and canceled by fuzzy logic system [2-8].

Since the idea of the fuzzy feedback linearization control based on Takagi-Sugeno (TS) models was presented in [2], various kinds of robust [7-8] and adaptive techniques [3-5] have been applied to the fuzzy feedback linearization control. While the adaptive fuzzy feedback linearization guarantees Lyapunov stability in the presence of uncertainty, it has some practical limitations due to its complex structures. From a practical point of view, robust approach is more suitable for the fuzzy feedback linearization to overcome the uncertainty [6-8]. The stability analysis was made in the frequency domain in [6] and the robust stability condition and design method using multivariable circle criterion have been presented in [7]. However, they based on graphical stability analysis, there exist some difficulties in being applied to the control problems directly.

On the other hand, The linear matrix inequality (LMI) theory is a new and fast growing field and a valuable alternative to the analytical method [9,10]. A variety of problems arising in system and control theory can be reduced to a few standard convex or quasiconvex optimization problems involving the LMI. Specifically, for a class of fuzzy control problems which is difficult to solve analytically, the LMI techniques can afford the practical solutions. In the recent papers [11-14], its applicability to the fuzzy control was shown clearly.

In order to obtain the numerical solutions for the fuzzy feedback linearization control systems, LMI based robust stability condition which can be solved numerically for the fuzzy feedback linearization regulator has been presented in [8]. However, the only stability analysis was done and the design problems were not handled. In addition, the transformation of the closed system into LMI form needed some complex procedure such as a loop transformation.

In this paper, we study a controller design as well as numerical stability analysis for the robust fuzzy feedback linearization control systems using TS fuzzy model. TS fuzzy model based control has been extensively studied up to now [11,12] because it can represent a nonlinear equation with a small number of rules [1].

To analyze the robust stability of the fuzzy feedback linearization control, we assume that the

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uncertainty is included in the model structure with known bounds. For these structured uncertainty, the L_2 robust stability of the closed system is analyzed by applying the LMI based convex optimization method. The stability problems are cast into diagonal norm bounded linear differential inclusions (DNLDI) and a generalized eigenvalue problem (GEVP). In the controller design part, based on the analysis methods, we present a systematic numerical method for finding the maximum stable ranges of the fuzzy feedback linearization control gains.

This paper is organized as follows. Section 2 discusses the fuzzy feedback linearization control scheme and in Section 3, the numerical stability analysis and design method are presented. The effectiveness of the proposed analysis and design scheme is illustrated through the detailed simulation, namely, the balancing of an inverted pendulum on a cart in Section 4. Finally concluding remarks are collected in Section 5.

2. PROBLEM FORMULATION

The fuzzy model represents a nonlinear system with the following form of fuzzy rules.

i-th plant rule:

IF x is M_{il} and $\dot{x}$ is M_{i2} and $\cdots$ and $x^{(n-1)}$ is M_{in}

Then $x^{(n)} = (\mathbf{a_i} + \mathbf{\Delta a_i}(t))^T \cdot \mathbf{x} + (b_i + \Delta b_i(t))u + d$, (1)

$i = 1, 2, 3, \cdots, r$

where $\mathbf{x} = [x, \dot{x}, \cdots, x^{(n-1)}]^T$ is the state vector which is assumed to be available and $\mathbf{a_i}$, $\mathbf{\Delta a_i}(t) \in R^{1\times n}$, b_i, $\Delta b_i(t) \in R$ and $d \in R$ denotes unknown external disturbance which belongs to L_2 space such that

$$\int_0^\infty d(t)^2 dt < \infty . \tag{2}$$

Also, M_{ij} is the fuzzy set and r is the number of fuzzy rules. Also, $\mathbf{\Delta a_i}(t)$ and $\Delta b_i(t)$ denote the norm-bounded time-varying modeling uncertainties for system and input matrices, respectively. The TS fuzzy model can be inferred as

$$x^{(n)} = \sum_{i=1}^{r} h_i(\mathbf{x})\{(\mathbf{a_i} + \mathbf{\Delta a_i}(t))^T \cdot x + (b_i + \Delta b_i(t))u\} + d, \tag{3}$$

where

$$w_i(\mathbf{x}) = \prod_{j=1}^{n} M_{ij}(x^{(j-1)}), \quad h_i(\mathbf{x}) = \frac{w_i(\mathbf{x})}{\sum_{i=1}^{r} w_i(\mathbf{x})} .$$

$M_{ij}(x^{(j-1)})$ is the grade of membership of $x^{(j-1)}$ in M_{ij}. It is assumed in this paper that

$$w_i(\mathbf{x}) \geq 0 \ , \ i = 1, 2, \cdots, r \ , \ \sum_{i=1}^{r} w_i(\mathbf{x}) > 0 \ .$$

Therefore,

$$h_i(\mathbf{x}) \geq 0 \ , \ i = 1, 2, \cdots, r \ , \ \sum_{i=1}^{r} h_i(\mathbf{x}) = 1 .$$

For (3) to be controllable, $\sum_{i=1}^{r} h_i(\mathbf{x}) b_i \neq 0$ for $\mathbf{x}$ in a certain controllability region $U_c \subset R^n$ is required. If this controllability requirement is satisfied and there is no uncertainty in (3), that is, $\mathbf{\Delta a_i} = 0$, $\Delta b_i = 0$, d=0,the following fuzzy feedback linearization controller (4) can cancel the nonlinearity of (3) and achieve exact linearization (5).

$$u = \frac{\sum_{i=1}^{r} h_i(\mathbf{x})(\mathbf{a}_d^T - \mathbf{a}_i^T) \cdot \mathbf{x}}{\sum_{i=1}^{r} h_i(\mathbf{x}) b_i}, \tag{4}$$

$$x^{(n)} = \mathbf{a}_d^T \cdot \mathbf{x} , \tag{5}$$

where we use the same $\mathbf{a_i}$, b_i and $h_i(\mathbf{x})$ with the fuzzy model (3) for all i and $\mathbf{a_d} \in R^n$ is chosen such that the exact linearized system (5) is asymptotically stable.

In practical application, however, uncertainty and disturbances are inevitable. Therefore, the exact linearization cannot be achieved.

Hence, for the robust stability, consider the following control law (6),

$$u = \frac{(\mathbf{a}_R^T + \sum_{i=1}^{r} h_i(\mathbf{x})(\mathbf{a}_d^T - \mathbf{a}_i^T)) \cdot x}{\sum_{i=1}^{r} h_i(\mathbf{x}) b_i}, \tag{6}$$

where $\mathbf{a_R} \in R^n$ is the appended input vector in order to reduce the disturbance, which comes from the uncertainties.

By substituting (6) into (3), the closed loop system can be written as (7).

$$\begin{aligned} x^{(n)} &= \mathbf{a}_d^T \cdot \mathbf{x} + \mathbf{a}_R^T \cdot \mathbf{x} + \sum_{i=1}^{r} h_i(\mathbf{x})\ \mathbf{\Delta a}_i(t)^T \cdot \mathbf{x} \\ &\quad + \frac{\sum_{i=1}^{r} h_i(\mathbf{x}) \Delta b_i(t)}{\sum_{i=1}^{r} h_i(\mathbf{x}) b_i} \{\sum_{i=1}^{r} h_i(\mathbf{x})(a_d + a_R - a_i)^T \cdot \mathbf{x}\} + d \\ &= \mathbf{a}_d^T \cdot \mathbf{x} + \mathbf{a}_N(t)^T \cdot \mathbf{x} + d \end{aligned} \tag{7}$$

where

$$a_N(t) = a_R + \sum_{i=1}^{r} h_i(x)\ \Delta a_i\ (t) \tag{8}$$

$$+\frac{\sum_{i=1}^{r} h_i(x)\Delta b_i(t)}{\sum_{i=1}^{r} h_i(x) b_i}\{\sum_{i=1}^{r} h_i(x)(a_d+a_R-a_i)\}.$$

In the next section, the robust stability analysis and the design of $\mathbf{a_R}$ for (7) is presented.

3. ROBUST STABILITY ANALYSIS AND DESIGN OF FEEDBACK LINEARIZATION CONTROL

3.1. Robust stability analysis

In order to give the numerical L_2 stability condition, the closed system (7) is cast into Diagonal Norm-bound Linear Differential Inclusions (DNLDI). DNLDI is a linear system with scalar, uncertain and time-varying feedback gains, each of which is bounded by one. The DNLDI formulation of the closed system (7) is given by

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bp} + \mathbf{w}, \quad \mathbf{p} = \mathbf{V(t)Cx}, \quad \mathbf{z} = \mathbf{Dx}, \tag{11}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & 1 \\ a_{d1} & a_{d2} & a_{d3} & \cdots & a_{dn} \end{bmatrix} \in R^{n\times n},$$

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ & & \vdots & & \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix} \in R^{n\times n},$$

$$\mathbf{C} = \begin{bmatrix} c_1 & 0 & 0 & \cdots & 0 \\ 0 & c_2 & 0 & \cdots & 0 \\ 0 & 0 & c_3 & \cdots & 0 \\ & & \vdots & & \\ 0 & 0 & 0 & \cdots & c_n \end{bmatrix} \in R^{n\times n}, \tag{12}$$

$$\mathbf{\Delta(t)} = \begin{bmatrix} \delta_1(t) & 0 & 0 & \cdots & 0 \\ 0 & \delta_2(t) & 0 & \cdots & 0 \\ 0 & 0 & \delta_3(t) & \cdots & 0 \\ & & \vdots & & \\ 0 & 0 & 0 & \cdots & \delta_n(t) \end{bmatrix},$$

$$\delta_j(t) = \begin{cases} \dfrac{a_{Nj}(t)}{c_j} & if\ c_j \neq 0 \\ 0 & if\ c_j = 0 \end{cases}$$

constraint :

$$\left| a_{Nj}(t) \right| \le c_j \ (\ i = 1, 2, \cdots, n) \tag{13}$$

or equivalently,

$$\mathbf{p^T p} \le \mathbf{x^T C^T Cx}, \tag{14}$$

$$\mathbf{D} = \mathbf{I} \in R^{n\times n}, \ \mathbf{p} \in R^n, \ \mathbf{z} \in R^n, \mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ d \end{bmatrix} \in R^n. \tag{15}$$

Remark 1: In (12), c_j ($j = 1,2,\cdots,n$) can be any non-negative real scalar satisfying the constraint (13) or **C** can be any diagonal positive semidefinite matrix satisfying the constraint (14). Note that c_j can be set to 0, only if there is no uncertainty in the corresponding a_j, i.e. $a_{Nj}(t) = 0$. In Appendix A, the selecting method of c_j ($j = 1,2,\cdots,n$) for the stability analysis is proposed.

In (15), $\mathbf{w}$ is the unknown external disturbance input which belongs to L_2 space such that

$$\int_0^\infty \mathbf{w}^T \mathbf{w}\, dt < \infty \tag{16}$$

and $\mathbf{z}$ is the output which is the same as the state $\mathbf{x}$.

Theorem 1 [9]: The system (11) is L_2 stable and its L_2 gain (10) is less than γ if there exist $\mathbf{P} > \mathbf{0}$ and $\tau \ge 0$ such that

$$\begin{bmatrix} \mathbf{A^T P} + \mathbf{PA} + \mathbf{D^T D} + \tau\mathbf{C^T C} & \mathbf{PB} & \mathbf{P} \\ \mathbf{B^T P} & -\tau\mathbf{I} & 0 \\ \mathbf{P} & 0 & -\gamma^2\mathbf{I} \end{bmatrix} \le \mathbf{0}. \tag{17}$$

Proof: Now, suppose there exist a quadratic function $V(\mathbf{x}) = \mathbf{x}^T\mathbf{Px}$, $\mathbf{P} > \mathbf{0}$, and $\gamma \ge 0$ such that for all t,

$$\begin{aligned} &\frac{d}{dt}V(\mathbf{x}) + \mathbf{z^T z} - \gamma^2\mathbf{w^T w} = \\ &\mathbf{x^T(A^T P + PA + D^T D)x} + 2\mathbf{x^T PBp} + 2\mathbf{Pw} - \gamma^2\mathbf{w^T w} \end{aligned} \tag{18}$$

for all $\mathbf{x}$ and $\mathbf{p}$ satisfying $\mathbf{p^T p} \le \mathbf{x^T C^T Cx}$.

Using the S-procedure of LMI techniques[9], (18) is equivalent to the existence of $\mathbf{P}$ and τ satisfying

$$\begin{bmatrix} \mathbf{A^T P} + \mathbf{PA} + \mathbf{D^T D} + \tau\mathbf{C^T C} & \mathbf{PB} & \mathbf{P} \\ \mathbf{B^T P} & -\tau\mathbf{I} & 0 \\ \mathbf{P} & \mathbf{0} & -\gamma^2\mathbf{I} \end{bmatrix} \le \mathbf{0}.$$

To show the L_2 gain (10) is less than γ, we integrate (18) from 0 to T, with the initial condition $\mathbf{x(0)} = \mathbf{0}$, to get

$$V(\mathbf{x}(T)) + \int_0^T (\mathbf{z^T z} - \gamma^2\mathbf{w^T w})dt \le 0. \tag{19}$$

Since $V(\mathbf{x}(T)) \geq 0$, this implies

$$\frac{\|\mathbf{z}\|_2}{\|\mathbf{w}\|_2} < \gamma . \tag{20}$$

□

Therefore, from the Theorem 1, we can obtain the upper bound on the L_2 gain by solving the following EigenValue Problem (EVP).

minimize γ

$\mathbf{P} > \mathbf{0}$, $\tau \geq 0$,

$$\begin{bmatrix} \mathbf{A}^T\mathbf{P} + \mathbf{PA} + \mathbf{D}^T\mathbf{D} + \tau\mathbf{C}^T\mathbf{C} & \mathbf{PB} & \mathbf{P} \\ \mathbf{B}^T\mathbf{P} & -\tau\mathbf{I} & \mathbf{0} \\ \mathbf{P} & \mathbf{0} & -\gamma^2\mathbf{I} \end{bmatrix} \leq \mathbf{0} . \tag{21}$$

Based on the Theorem 1, the analysis procedure can be summarized as follows.

Step 1: Cast the closed system (7) into DNLDI (11).

Step 2: Select c_j (j=1,2,$\cdots$,n) as in Appendix A.

Step 3: Check the stability condition of Theorem 1. This can be easily done by solving the feasibility problem.

Step 4: If there exists a feasible EVP solution $\gamma_{\min}$, then the closed system is robust stable in L_2 sense and L_2 gain is less than $\gamma_{\min}$.

Also, we can easily extend the derived input-output stability condition to Lyapunov stability for the unforced system by the following lemma.

Lemma 1: **x**=**0** is a globally attractive equilibrium of the unforced system of the closed loop system (7) (i.e., d=0) if there exist $\mathbf{P} > \mathbf{0}$ and $\tau \geq 0$ which satisfy LMI (22).

$$\begin{bmatrix} \mathbf{A}^T\mathbf{P} + \mathbf{PA} + \mathbf{D}^T\mathbf{D} + \tau\mathbf{C}^T\mathbf{C} & \mathbf{PB} & \mathbf{P} \\ \mathbf{B}^T\mathbf{P} & -\tau\mathbf{I} & \mathbf{0} \\ \mathbf{P} & \mathbf{0} & -\gamma^2\mathbf{I} \end{bmatrix} \leq \mathbf{0} \tag{22}$$

Proof: The proof of this lemma will be given in Appendix B.

3.2. Robust stable design

Our problem is that of determining the L_2 robust stability range of a_{Rj} ($j = 1,2,\cdots,n$) which can maintain the L_2 gain of the closed system (7) within the specified upper bound $\gamma_{\max}$. From the constraint (13), c_j can be regarded as the upper bound on $|a_{Nj}(t)|$ (j=1,2,$\cdots$,n) which was derived in Appendix A.

Therefore, in order to determine a robust stable range on a_{Rj}, we need to find the largest possible c_j for which Theorem 1 holds with $\gamma = \gamma_{\max}$ should be obtained. This can be obtained by solving the following optimization problem (23).

maximize $c_1, c_2, \cdots, c_n$

subject to

$\mathbf{P} > \mathbf{0}$, $\tau \geq 0$,

$$\begin{bmatrix} \mathbf{A}^T\mathbf{P} + \mathbf{PA} + \mathbf{D}^T\mathbf{D} + \tau\mathbf{C}^T\mathbf{C} & \mathbf{PB} & \mathbf{P} \\ \mathbf{B}^T\mathbf{P} & -\tau\mathbf{I} & \mathbf{0} \\ \mathbf{P} & \mathbf{0} & -\gamma_{\max}{}^2\mathbf{I} \end{bmatrix} \leq \mathbf{0} . \tag{23}$$

However, it is difficult to solve the multiple parameter optimization problem (23) straightforward. Instead, by splitting (23) into the single parameter optimization problems (24) for each i, it is possible to derive the feasible solution of (23) from the solutions of (24).

maximize c_i

subject to

$\mathbf{P_i} > \mathbf{0}$, $\tau_i \geq 0$,

$$\begin{bmatrix} \mathbf{A}^T\mathbf{P_i} + \mathbf{P_i}\mathbf{A} + \mathbf{D}^T\mathbf{D} + \tau_i\mathbf{C_i}^T\mathbf{C_i} & \mathbf{P_i}\mathbf{B} & \mathbf{P}_i \\ \mathbf{B}^T\mathbf{P}_i & -\tau_i\mathbf{I} & \mathbf{0} \\ \mathbf{P_i} & \mathbf{0} & -\gamma_{\max}^2\mathbf{I} \end{bmatrix} \leq 0, \tag{24}$$

where $\mathbf{C}_i = diag(0,\cdots,0,c_i,0,\cdots,0)$

If we define $\lambda_i = c_i^2$, the optimization problem (24) can be viewed as the Generalized Eigen-Value Problem (GEVP) (25).

maximize λ_i

subject to

$\mathbf{P_i} > \mathbf{0}$, $\tau_i \geq 0$, $\lambda_i \geq 0$

$$\begin{bmatrix} \mathbf{A}^T\mathbf{P_i} + \mathbf{P_i}\mathbf{A} + \mathbf{D}^T\mathbf{D} + \tau_i\mathbf{C_i}^T\mathbf{C_i} & \mathbf{P_i}\mathbf{B} & \mathbf{P}_i \\ \mathbf{B}^T\mathbf{P}_i & -\tau_i\mathbf{I} & \mathbf{0} \\ \mathbf{P_i} & \mathbf{0} & -\gamma_{\max}^2\mathbf{I} \end{bmatrix} \leq 0, \tag{25}$$

where $\mathbf{E_i} = \dfrac{\mathbf{C_i}}{c_i}$. Thus, the above GEVP can be easily solved by well-established LMI optimization techniques [10].

Denote the solutions of GEVP (25) as $\overline{\lambda}$ $(i = 1,2,\cdots,n)$. Then, the solutions of the optimization problem (24) can be written as $\overline{c_i} = \sqrt{\overline{\lambda_i}}$ (i=1,2,$\cdots$,n). Now, it should be noted that $\overline{c_1}, \overline{c_2}, \cdots, \overline{c_n}$ can not be a feasible solution of the optimization problem (23).

$$\text{For, } \overline{C} = \begin{bmatrix} \overline{c_1} & 0 & 0 & \cdots & 0 \\ 0 & \overline{c_2} & 0 & \cdots & 0 \\ 0 & 0 & \overline{c_3} & \cdots & 0 \\ & & \vdots & & \\ 0 & 0 & 0 & \cdots & \overline{c_n} \end{bmatrix} = \sum_{i=1}^{n} \overline{C_i} , \tag{26}$$

where $\overline{\mathbf{C}}_i = diag(0, \cdots, 0, \overline{c}_i, 0, \cdots, 0)$ $\leftarrow i\text{-}1 \rightarrow \leftarrow n\text{-}i \rightarrow$, it can not be guaranteed (17) holds. Thus, some modifications are needed to obtain a feasible solution. The modified $\overline{C^m}$ can be written as

$$\overline{\mathbf{C}^m} = \begin{bmatrix} \overline{c_1^m} & 0 & 0 & \cdots & 0 \\ 0 & \overline{c_2^m} & 0 & \cdots & 0 \\ 0 & 0 & \overline{c_3^m} & \cdots & 0 \\ & & \vdots & & \\ 0 & 0 & 0 & \cdots & \overline{c_n^m} \end{bmatrix}$$

$$= \frac{1}{\sqrt{\sum_{i=1}^{n}\overline{\tau}_i}} \begin{bmatrix} \sqrt{\overline{\tau}_1}\,\overline{c_1} & 0 & 0 & \cdots & 0 \\ 0 & \sqrt{\overline{\tau}_2}\,\overline{c_2} & 0 & \cdots & 0 \\ 0 & 0 & \sqrt{\overline{\tau}_2}\,\overline{c_2} & \cdots & 0 \\ & & \vdots & & \\ 0 & 0 & 0 & \cdots & \sqrt{\overline{\tau}_n}\,\overline{c_n} \end{bmatrix} \quad (27)$$

$$= \frac{\sum_{i=1}^{n}(\sqrt{\overline{\tau}_i}\,\overline{C_i})}{\sqrt{\sum_{i=1}^{n}\overline{\tau}_i}},$$

where $\overline{\tau}_i$ denotes τ_i corresponding to $\overline{\lambda}_i$ or $\overline{c}_i$ (i=1,2,$\cdots$,n).

In Theorem 2, it is shown that Theorem 1 holds for $\overline{\mathbf{C}^m}$ in (27).

Theorem 2: For $\overline{\mathbf{C}^m}$ in (27), there exists $\mathbf{P} > \mathbf{0}$ and $\tau \geq 0$ which satisfy the LMI L_2 stability condition as.

$$\begin{bmatrix} A^T\mathbf{P} + \mathbf{PA} + \mathbf{D}^T\mathbf{D} + \tau\overline{\mathbf{C}^m}^T\overline{\mathbf{C}^m} & \mathbf{PB} & \mathbf{P} \\ \mathbf{B}^T\mathbf{P} & -\tau\,\mathbf{I} & \mathbf{0} \\ \mathbf{P} & \mathbf{0} & -\gamma_{\max}^2\mathbf{I} \end{bmatrix} \leq \mathbf{0}. \quad (28)$$

Proof: Since $\overline{c}_i$ (i=1,2,$\cdots$,n) is the solution of the optimization problem (24), the following holds for all i

$$\begin{bmatrix} \mathbf{A}^T\overline{\mathbf{P}_i} + \overline{\mathbf{P}_i}\mathbf{A} + \mathbf{D}^T\mathbf{D} + \overline{\tau}_i\overline{\mathbf{C}_i}^T\overline{\mathbf{C}_i} & \overline{\mathbf{P}_i}\mathbf{B} & \overline{\mathbf{P}_i} \\ \mathbf{B}^T\overline{\mathbf{P}_i} & -\overline{\tau}_i\mathbf{I} & \mathbf{0} \\ \overline{\mathbf{P}_i} & \mathbf{0} & -\gamma_{\max}^2\mathbf{I} \end{bmatrix} \leq \mathbf{0}, \quad (29)$$

where $\overline{\mathbf{P}_i}$ denotes $\mathbf{P}_i$ corresponding to $\overline{\lambda}_i$ or $\overline{c}_i$ (i= 1,2,$\cdots$,n).

Hence, from the property of the negative semi-definite matrix (30) also holds.

$$\frac{1}{n}\sum_{i=1}^{n}\begin{bmatrix} \mathbf{A}^T\overline{\mathbf{P}_i} + \overline{\mathbf{P}_i}\mathbf{A} + \mathbf{D}^T\mathbf{D} + \overline{\tau}_i\overline{\mathbf{C}_i}^T\overline{\mathbf{C}_i} & \overline{\mathbf{P}_i}\mathbf{B} & \overline{\mathbf{P}_i} \\ \mathbf{B}^T\overline{\mathbf{P}_i} & -\overline{\tau}_i\mathbf{I} & \mathbf{0} \\ \overline{\mathbf{P}_i} & \mathbf{0} & -\gamma_{\max}^2\mathbf{I} \end{bmatrix} \leq \mathbf{0}. \quad (30)$$

By rearranging the summations, (30) becomes

$$\begin{bmatrix} \mathbf{A}^T(\sum_{i=1}^{n}\frac{\overline{\mathbf{P}_i}}{n}) + (\sum_{i=1}^{n}\frac{\overline{\mathbf{P}_i}}{n})\mathbf{A} + \mathbf{D}^T\mathbf{D} + (\sum_{i=1}^{n}\frac{\overline{\tau}_i}{n})\overline{\mathbf{C}_i}^T\overline{\mathbf{C}_i} & (\sum_{i=1}^{n}\frac{\overline{\mathbf{P}_i}}{n})\mathbf{B} & (\sum_{i=1}^{n}\frac{\overline{\mathbf{P}_i}}{n}) \\ \mathbf{B}^T(\sum_{i=1}^{n}\frac{\overline{\mathbf{P}_i}}{n}) & -(\sum_{i=1}^{n}\frac{\overline{\tau}_i}{n})\mathbf{I} & \mathbf{0} \\ (\sum_{i=1}^{n}\frac{\overline{\mathbf{P}_i}}{n}) & \mathbf{0} & -\gamma_{\max}^2\mathbf{I} \end{bmatrix} \leq \mathbf{0}. \quad (31)$$

Using the property of $\overline{\mathbf{C}_i}$, it can be easily shown that

$$\sum_{i=1}^{n}\frac{\overline{\tau_i}}{n}\overline{\mathbf{C}_i}^T\overline{\mathbf{C}_i} = (\sum_{i=1}^{n}\sqrt{\frac{\overline{\tau_i}}{n}}\overline{\mathbf{C}_i}^T)(\sum_{i=1}^{n}\sqrt{\frac{\overline{\tau_i}}{n}}\overline{\mathbf{C}_i}) \quad (32)$$

holds.

Employing (31), (32) can be written as

$$\begin{bmatrix} \mathbf{A}^T(\sum_{i=1}^{n}\frac{\overline{\mathbf{P}_i}}{n}) + (\sum_{i=1}^{n}\frac{\overline{\mathbf{P}_i}}{n})\mathbf{A} + \mathbf{D}^T\mathbf{D} + (\sum_{i=1}^{n}\sqrt{\frac{\overline{\tau_i}}{n}}\overline{\mathbf{C}_i}^T)(\sum_{i=1}^{n}\sqrt{\frac{\overline{\tau_i}}{n}}\overline{\mathbf{C}_i}) & (\sum_{i=1}^{n}\frac{\overline{\mathbf{P}_i}}{n})\mathbf{B} & (\sum_{i=1}^{n}\frac{\overline{\mathbf{P}_i}}{n}) \\ \mathbf{B}^T(\sum_{i=1}^{n}\frac{\overline{\mathbf{P}_i}}{n}) & -(\sum_{i=1}^{n}\frac{\overline{\tau_i}}{n})\mathbf{I} & \mathbf{0} \\ (\sum_{i=1}^{n}\frac{\overline{\mathbf{P}_i}}{n}) & \mathbf{0} & (-\gamma_{\max}^2\mathbf{I}) \end{bmatrix}$$

$$\leq \mathbf{0}. \quad (33)$$

Let us choose

$$\mathbf{P} = \sum_{i=1}^{n}\frac{\overline{\mathbf{P}_i}}{n} \text{ and } \tau = \sum_{i=1}^{n}\frac{\overline{\tau_i}}{n}. \quad (34)$$

Using (27), (34) and (33) can be expressed as

$$\begin{bmatrix} \mathbf{A}^T\mathbf{P} + \mathbf{PA} + \mathbf{D}^T\mathbf{D} + \tau\overline{\mathbf{C}^m}^T\overline{\mathbf{C}^m} & \mathbf{PB} & \mathbf{P} \\ \mathbf{B}^T\mathbf{P} & -\tau\mathbf{I} & \mathbf{0} \\ \mathbf{P} & \mathbf{0} & -\gamma_{\max}^2\mathbf{I} \end{bmatrix} \leq \mathbf{0}\cdot \quad (35)$$

Therefore, for $\overline{\mathbf{C}^m}$ in (27), there exists **P>0** and $\tau \geq 0$ which satisfies LMI L_2 stability condition (28). □

Since Theorem 2 holds for $\overline{\mathbf{C}^m}$ in (27), $\overline{c_1^m}, \overline{c_2^m}, \cdots, \overline{c_n^m}$ can be a feasible solution of the optimization problem (23). Therefore $\overline{c_j^m}$ (j = 1,2,$\cdots$,n) can be used as the largest possible $\overline{c_i}$ ($i = 1, 2, \cdots, n$) for which Theorem 1 holds.

Thus, using the admissible bounds of $|a_{Nj}(t)|$ with respect to a_{Rj}, the robust stable range of a_{Rj} can be expressed by the following set representation (36).

$$\left\{ a_{Rj} \;\middle|\; \begin{array}{l} |a_{Rj}| + \max_i |\Delta a_{ij}(t)| \\ + \dfrac{\max_i |\Delta b_i(t)|}{\min_i |b_i|}(\max_i |a_{dj} + a_{Rj} - a_{ij}|) \leq c_j^m \end{array} \right\} \quad (36)$$

$$j = 1, 2, \cdots, n$$

The control design procedure is summarized as follows.

Step 1: Cast the closed loop system (7) into DNLDI (11).

Step 2: Solve the GEVP (25).

Step 3: Find the stable range (36) of a_{Rj} from $\overline{\mathbf{C}^m}$ in (27)

Step 4: Select proper a_{Rj} in the set (36).

4. SIMULATIONS

Consider the problem of balancing and swing-up of an inverted pendulum on a cart shown in Fig. 1. The equations of motion [3] for the pendulum are

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = f(\mathbf{x}) + g(\mathbf{x})u + d(t) \quad (37)$$
$$= \frac{g\sin(x_1) - amlx_2^2\sin(2x_1)/2 - a\cos(x_1)u}{4l/3 - aml\cos^2(x_1)} + d(t),$$

where $\mathbf{x} = [x_1 \quad x_2]^T$ and x_1 denotes the angle (in radians) of the pendulum from the vertical, and x_2 is the angular velocity. g=9.8m/s^2 is the gravity constant, m is the mass of the pendulum, M is the mass of the cart, 2l is the length of the pendulum, u is the control force applied to the cart (in Newtons). $d(t)$ is the external disturbance and $a = \frac{1}{m+M}$. We choose, m = 2.0kg, M = 8.0kg and 2l = 1.0m in the simulation.

The dynamic equations (37) can be approximated by the following two fuzzy rules [11] and the membership functions used in this fuzzy model are shown in Fig. 2.

$$Rule\ 1: IF\ x\ is\ about\ 0$$
$$THEN\ \ddot{x} = (\mathbf{a_1} + \mathbf{\Delta a_1}(t))^T \cdot \mathbf{x} + (b_1 + \Delta b_1(t))u + d$$
$$Rule\ 2: IF\ x\ is\ about\ \pm\frac{\pi}{2}\ (|x| < \frac{\pi}{2}) \quad (38)$$
$$THEN\ \ddot{x} = (\mathbf{a_2} + \mathbf{\Delta a_2}((t)^T \cdot \mathbf{x} + (b_2 + \Delta b_2((t)u + d$$

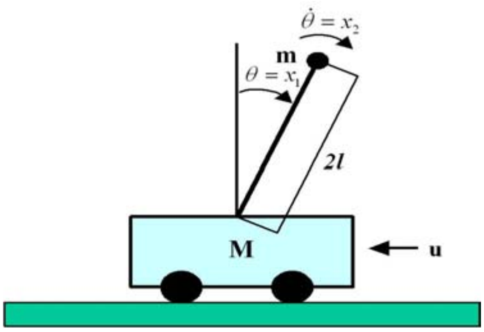

Fig. 1. The inverted pendulum system.

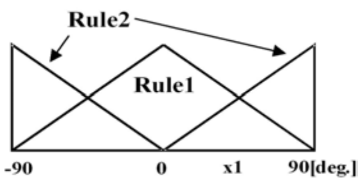

Fig. 2. Membership functions.

(38) can be inferred as

$$\ddot{x} = \sum_{i=1}^{2} h_i(\mathbf{x})\left\{(\mathbf{a_i} + \mathbf{\Delta a_i}(t))^T \cdot x + (b_i + \Delta b_i(t))u\right\} + d, \quad (39)$$

where $w_i(\mathbf{x}) = \prod_{j=1}^{2} M_{ij}(x^{(j-1)})$, $h_i(\mathbf{x}) = \frac{w_i(\mathbf{x})}{\sum_{i=1}^{2} w_i(\mathbf{x})}$ and

$$\mathbf{a_1} = \left[\frac{g}{4l/3 - aml} \quad 0\right] = [17.29 \quad 0],$$
$$\mathbf{a_2} = \left[\frac{2g}{\pi(4l/3 - aml\beta^2)} \quad 0\right] = [9.35 \quad 0],$$
$$b_1 = -\frac{a}{4l/3 - aml} = -0.1765,$$
$$b_1 = -\frac{a\beta}{4l/3 - aml\beta^2} = -0.0052.$$

We assume that $\mathbf{\Delta a_1}$, $\mathbf{\Delta a_2}$, Δb_1, Δb_2 are unknown but bounded as follows:

$-1 \le \Delta a_{11} \le 1$, $-0.5 \le \Delta a_{12} \le 0.5$,
$-1 \le \Delta a_{21} \le 1$, $-0.5 \le \Delta a_{22} \le 0.5$,
$-0.001 \le \Delta b_1 \le 0.001$, $\quad -0.001 \le \Delta b_2 \le 0.001$.

In the following analysis and design section, we use the feedback linearization control law as

$$u = \frac{(\mathbf{a_R}^T + \sum_{i=1}^{r} h_2(\mathbf{x})(\mathbf{a_d}^T - \mathbf{a_i}^T)) \cdot x}{\sum_{i=1}^{r} h_2(\mathbf{x}) b_i} \quad (40)$$

and then, the closed loop system by substituting (40) into (39) yields

$$\ddot{x} = \mathbf{a}_d^T \cdot x + \mathbf{a}_N(t)^T \cdot \mathbf{x} + d, \quad (41)$$

where $\mathbf{a}_N(t) = \mathbf{a_R} + \sum_{i=1}^{r} h_i(\mathbf{x})\ \mathbf{\Delta a_i}(t)$

$$+ \frac{\sum_{i=1}^{r} h_i(\mathbf{x})\Delta b_i(t)}{\sum_{i=1}^{r} h_i(\mathbf{x}) b_i}\{\sum_{i=1}^{r} h_i(\mathbf{x})(\mathbf{a_d} + \mathbf{a_R} - \mathbf{a_i})\}.$$

4.1. Robust stability analysis

The robust stability of the feedback linearization control system (41) with $\mathbf{a_d}$ = [-1 -1] and $\mathbf{a_R}$ = [-3 -3] is analyzed.

Step 1: Represent the closed system (41) into the DNLDI (42).

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bp} + \mathbf{w},\ \mathbf{p} = \mathbf{\Delta(t)Cx},\ \mathbf{z} = \mathbf{Dx}, \quad (42)$$

where

$$\mathbf{A}=\begin{bmatrix}0 & 1\\ -1 & -1\end{bmatrix},\mathbf{B}=\begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix},\mathbf{C}=\begin{bmatrix}c_1 & 0\\ 0 & c_2\end{bmatrix}, \quad (43)$$

$$\mathbf{\Delta(t)}=\begin{bmatrix}\delta_1(t) & 0\\ 0 & \delta_2(t)\end{bmatrix},\delta_j(t)=\begin{cases}\dfrac{a_{Nj}(t)}{c_j} & if\ c_j\neq 0\\ 0 & if\ c_j=0.\end{cases}$$

constraint: $|a_{Nj}(t)|\le c_j$ (i=1,2) or equivalently, $\mathbf{P}^T\mathbf{P}\le \mathbf{x}^T\mathbf{C}^T\mathbf{C}\mathbf{x}$.

Step 2: Select c_j $(i=1,2)$ by

$$c_1=|a_{R1}|+\max_i|\Delta a_{i1}(t)|+\frac{\max_i|\Delta b_i(t)|}{\min_i|b_i|}$$

$$(\max_i|a_{d1}+a_{R1}-a_{i1}|)=8.088,$$

$$c_2=|a_{R2}|+\max_i|\Delta a_{i2}(t)|+\frac{\max_i|\Delta b_i(t)|}{\min_i|b_i|}$$

$$(\max_i|a_{d2}+a_{R2}-a_{i2}|)=4.268.$$

Step 3: Solve the following GEVP (44) using the interior point method of LMI techniques [10,15].

Minimize λ

$$\mathbf{P}>\mathbf{0},\ \tau\ge 0,$$
$$\begin{bmatrix}\mathbf{A}^T\mathbf{P}+\mathbf{PA}+\mathbf{D}^T\mathbf{D}+\tau\mathbf{C}^T\mathbf{C} & \mathbf{PB} & \mathbf{P}\\ \mathbf{B}^T\mathbf{P} & -\tau\mathbf{I} & \mathbf{0}\\ \mathbf{P} & \mathbf{0} & -\gamma^2\mathbf{I}\end{bmatrix}\le\mathbf{0}. \quad (44)$$

As a feasible GEVP solution, we obtain

$$\gamma_{\min}=0.0381 \text{ with } \tau=0.0231 \text{ and}$$
$$P=\begin{bmatrix}5.0103 & 10.1919\\ 10.1919 & 21.4886\end{bmatrix}. \quad (45)$$

Step 4: Since there exists a feasible EVP solution $\gamma_{\min}>0$, the closed loop system (41) is robust stable in L_2 sense and L_2 gain is less than $\gamma_{\min}=0.0381$.

4.1. Robust stable design

Consider the design problem for a_{Rj}, j=1,2, for the feedback linearization control system (41) with $\mathbf{a_d}$= [-1 -1].

Step 1: Cast the closed loop system into DNLDI.

This step is the same as step 1 in the analysis part.

Step 2: Solve the GEVP (25) for $j=1,2$ Using the GEVP solver [15], we have

For j=1 : $\overline{\lambda_1}=133.3332$, $\overline{c_1}=11.547$, $\overline{\tau_1}=0.0015$,

$$\overline{\mathbf{P}}_1=\begin{bmatrix}4.2965 & 0.6127\\ 0.6127 & 5.4184\end{bmatrix}. \quad (46)$$

For j=2 : $\overline{\lambda_1}=555.5556$, $\overline{c_1}=23.5702$, $\overline{\tau_1}=0.0015$,

$$\overline{\mathbf{P}}_2=\begin{bmatrix}5.4246 & 3.7481\\ 3.7481 & 4.3813\end{bmatrix}. \quad (47)$$

where $\gamma=\gamma_{\max}=0.01$ was specified.

Step 3: Find the stable range of a_{Rj} from $\overline{\mathbf{C_m}}$.

Compute $\overline{\mathbf{C_m}}$ as

$$\overline{\mathbf{C^m}}=\begin{bmatrix}\bar{C}_1^m & 0\\ 0 & \bar{C}_2^m\end{bmatrix}=\frac{1}{\sqrt{\sum_{j-1}^{2}\bar{\tau}_j}}\begin{bmatrix}\sqrt{\bar{\tau}_1}\bar{c}_1 & 0\\ 0 & \sqrt{\bar{\tau}_2}\bar{c}_2\end{bmatrix}$$
$$=\begin{bmatrix}8.165 & 0\\ 0 & 16.667\end{bmatrix}. \quad (48)$$

Therefore, the robust stable ranges of can be expressed by (49).

$$\left\{a_{R1}\,\middle|\,|a_{R1}|+\max_i|\Delta a_{i1}(t)|+\frac{\max_i|\Delta b_i(t)|}{\min_i|b_i|}.(\max_i|a_{d1}+a_{R1}-a_{i1}|)\le 8.165\right\}$$
$$\left\{a_{R2}\,\middle|\,|a_{R2}|+\max_i|\Delta a_{i2}(t)|+\frac{\max_i|\Delta b_i(t)|}{\min_i|b_i|}.(\max_i|a_{d2}+a_{R2}-a_{i2}|)\le 16.667\right\} \quad (49)$$

Step 4: Select proper a_{Rj} in the set (49).

Fig. 3 shows the region of a_{R1} and a_{R2} from the obtained (49), where we choose the parameters as a_{R1}= -2.5 and a_{R2}= -8.

In the computer simulation, as a disturbance $d(t)$ which belongs to L_2 space, the signal shown in Fig. 4 is used. Figs. 5 and 6 illustrate the simulation results in which the initial condition is zero. In Figs. 7 and 8, the energy of the disturbance and the output are plotted with respect to time, respectively.

L_2 norm of the input and output can be computed as

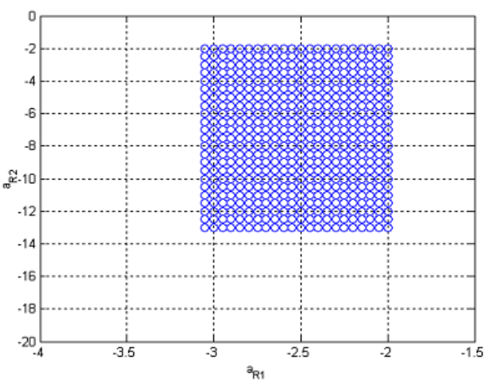

Fig. 3. Region of a_{R1} and a_{R2}.

$$\|\mathbf{w}\|_2 = \int_0^\infty \mathbf{w}^T\mathbf{w}\, dt = \int_0^\infty d(t)^2\, dt = 10\ , \tag{50}$$

$$\|\mathbf{z}\|_2 = \int_0^\infty \mathbf{z}^T\mathbf{z}\, dt = \int_0^\infty \mathbf{x}^T\mathbf{x}\, dt = 0.0858\ . \tag{51}$$

Thus L_2 gain is

$$\sup_{\|\mathbf{w}\|_2 \neq 0} \frac{\|\mathbf{z}\|_2}{\|\mathbf{w}\|_2} = 0.00858. \tag{52}$$

The simulation results illustrate that the closed system (41) is robust stable in L_2 sense and L_2 gain is less than 0.01 which is specified in the design procedure, Step 2.

Also, in order to analyze the Lyapunov stability, the simulation results for the unforced system, i.e. $d(t) = 0$ and the initial condition $\mathbf{x}_0 = [1 \quad 0]$ are presented in Figs. 9 and 10.

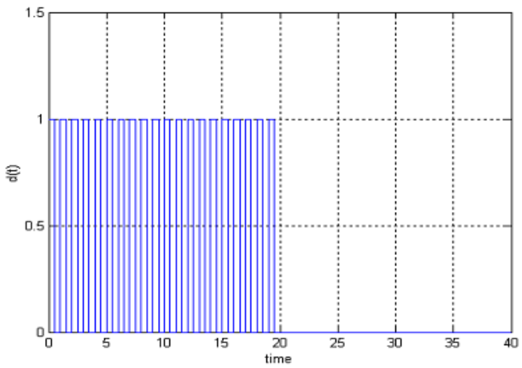

Fig. 4. External disturbance.

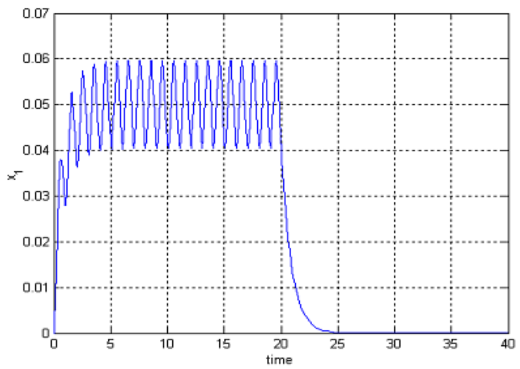

Fig. 5. Simulation result of state.

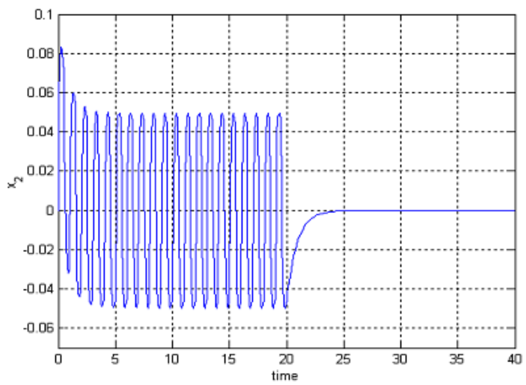

Fig. 6. Simulation result of state.

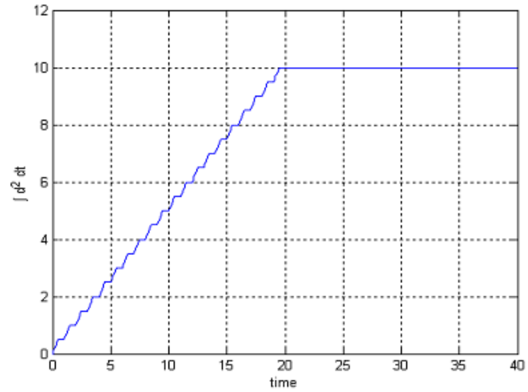

Fig. 7. Time variation of external disturbance energy.

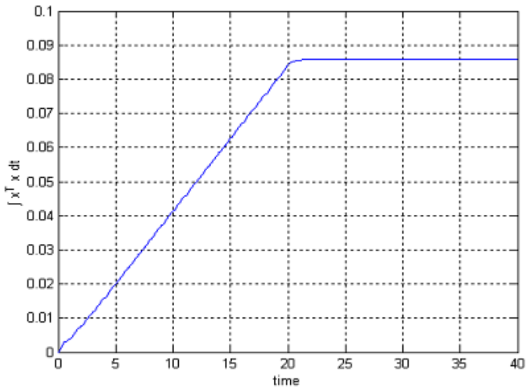

Fig. 8. Time variation of output energy.

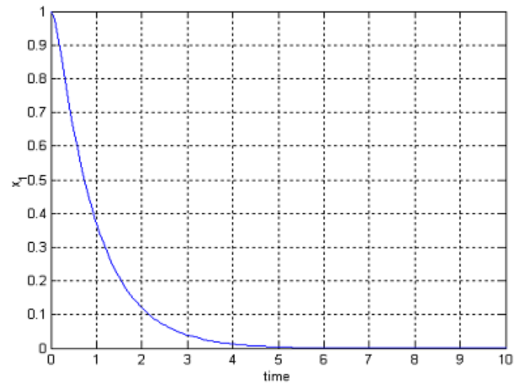

Fig. 9. Simulation result of state x_1 (unforced system).

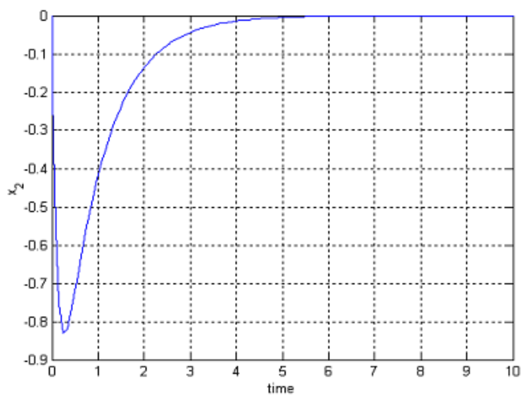

Fig. 10. Simulation result of state x_2 (unforced system).

5. CONCLUSIONS

In this work, we have presented the LMI-based L_2 robust stability analysis and design method for the fuzzy feedback linearization control systems. The plant was represented by well-known TS fuzzy model and the analysis and design problems was numerically solved by casting the closed loop system into DNLDI and GEVP form. In the examples, the fuzzy feedback linearization controller was developed efficiently and the validity of the proposed analysis and design scheme was shown.

APPENDIX A

Although c_j $(j = 1, 2, \cdots, n)$ can be any positive real scalar satisfying the constraint (13), c_j $(j = 1, 2, \cdots, n)$ should be chosen as the minimum upper bound for $|a_{Nj}(t)|$ to avoid the conservative analysis. In order to obtain the minimum upper bound for $|a_{Nj}(t)|$, (8) is written in the component form as in (A.1).

$$a_{Nj}(t) = a_{Rj} + \sum_{i=1}^{r} h_i(\mathbf{x}(t))\Delta a_{ij}(t) + \frac{\sum_{i=1}^{r} h_i(\mathbf{x}(t))\Delta b_i(t)}{\sum_{i=1}^{r} h_i(\mathbf{x}(t))b_i} \sum_{i=1}^{r} h_i(\mathbf{x}(t))(a_{dj} + a_{Rj} - a_{ij}), \quad (j = 1, 2, \cdots, n). \tag{A.1}$$

Then, the following inequality (A.2) holds for all j in which we used basic assumption,

$$\sum_{i=1}^{r} h_i(\mathbf{x}(t)) = 1 \text{ and } \max_x h_i(\mathbf{x}(t)) = 1 .$$

$$|a_{Nj}(t)| \le |a_{Rj}| + \left|\sum_{i=1}^{r} h_i(\mathbf{x}(t))\Delta a_{ij}(t)\right| + \left| \frac{\sum_{i=1}^{r} h_i(\mathbf{x}(t))\Delta b_i(t)}{\sum_{i=1}^{r} h_i(\mathbf{x}(t))b_i} \sum_{i=1}^{r} h_i(\mathbf{x}(t))(a_{dj} + a_{Rj} - a_{ij}) \right| . \tag{A.2}$$

The second and third terms in the right side of (A.2) satisfy (A.3) and (A.4).

$$\left|\sum_{i=1}^{r} h_i(\mathbf{x}(t))\Delta a_{ij}(t)\right| \le \max_i |\Delta a_{ij}(t)| \tag{A.3}$$

$$\left| \frac{\sum_{i=1}^{r} h_i(\mathbf{x}(t))\Delta b_i(t)}{\sum_{i=1}^{r} h_i(\mathbf{x}(t))b_i} \sum_{i=1}^{r} h_i(\mathbf{x}(t))(a_{dj} + a_{Rj} - a_{ij}) \right| \le \frac{\max_i |\Delta b_i(t)|}{\max_i |b_i|} (\max_i |a_{dj} + a_{Rj} - a_{ij}|) \tag{A.4}$$

Then, the following inequality holds for all j .

$$|a_{Nj}(t)| \le |a_{Rj}| + \max_i |\Delta a_{ij}(t)| + \frac{\max_i |\Delta b_i(t)|}{\max_i |b_i|} (\max_i |a_{dj} + a_{Rj} - a_{ij}|) .$$

Therefore, we choose

$$C_j = |a_{Rj}| + \max_i |\Delta a_{ij}(t)| + \frac{\max_i |\Delta b_i(t)|}{\max_i |b_i|} (\max_i |a_{dj} + a_{Rj} - a_{ij}|), \quad j = 1, 2, \cdots, n \tag{A.5}$$

for less conservative stability analysis.

APPENDIX B

To prove Lemma 1, we need the following Theorem.

Theorem 3 [16]**:** Consider the system,

$$\dot{\mathbf{x}} = \mathbf{Ax}(t) + \mathbf{Be}(t), \quad \mathbf{y}(t) = \mathbf{Cx}(t) , \quad \mathbf{e}(t) = \mathbf{u}(t) - \Phi[t, \mathbf{y}(t)] , \tag{B.1}$$

where $\mathbf{x}(t) \in R^n$, $\mathbf{u}(t) \in R^m$, $\mathbf{y}(t) \in R^l$ and $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are matrices of compatible dimensions and $\Phi : R_+ \times R^l \to R^m$ satisfies $\Phi(t, \mathbf{0}) = \mathbf{0}, \forall t \ge 0$. If the following three conditions are satisfied, $\mathbf{x} = 0$ is a globally equilibrium of the unforced system.

i) is globally Lipschitz continuous; i.e., there exists a finite constant μ such that $\|\Phi(t, \mathbf{y}_1) - \Phi(t, \mathbf{y}_2)\| \le \mu \|\mathbf{y}_1 - \mathbf{y}_2\|, \forall t \ge 0, \forall \mathbf{y}_1, \mathbf{y}_2 \in R^l$

ii) the pair **(A,B)** is controllable, and the pair **(C,A)** is observable.

iii) the forced system is L_2 stable.

Proof: Proof of this theorem can be found in [16].

In order to prove Lemma 1, the closed loop system (7) is expressed as (B.1), where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ & & \vdots & & \\ a_{d1} & a_{d2} & a_{d3} & \cdots & a_{dn} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ & & \vdots & & \\ -1 & -1 & -1 & \cdots & -1 \end{bmatrix}, \tag{B.2}$$

$$\mathbf{C}=\begin{bmatrix}1&0&0&\cdots&0\\0&1&0&\cdots&0\\0&0&1&\cdots&0\\&&\vdots&&\\0&0&0&\cdots&1\end{bmatrix},$$

$$\mathbf{\Phi}(t)=\begin{bmatrix}a_{N1}(t)&0&0&\cdots&0\\0&a_{N2}(t)&0&\cdots&0\\0&0&a_{N3}(t)&\cdots&0\\&&\vdots&&\\0&0&0&\cdots&a_{Nn}(t)\end{bmatrix},\mathbf{u}=\begin{bmatrix}0\\0\\0\\\vdots\\-d\end{bmatrix}.$$

Then, since a_{Nj} is bounded for all j and t, we can assume that $\|\mathbf{\Phi}(t)\|\le\mu$ for all t, where μ is a finite constant. Therefore, the following inequality holds for all $t\ge 0$ and for all $\mathbf{y_1},\mathbf{y_2}$.

$$\begin{aligned}&\|\mathbf{\Phi}(t)\mathbf{y_1}-\mathbf{\Phi}(t)\mathbf{y_2}\|=\|\mathbf{\Phi}(t)(\mathbf{y_1}-\mathbf{y_2})\|\\&\le\|\mathbf{\Phi}(t)\|\|\mathbf{y_1}-\mathbf{y_2}\|\le\mu\|\mathbf{y_1}-\mathbf{y_2}\|.\end{aligned}\quad\text{(B.3)}$$

Therefore, $\mathbf{\Phi}(t)$ is globally Lipschitz continuous and the pair (**A**,**B**) and the pair (**C**,**A**) can be easily shown to be controllable and observable respectively, independent of $\mathbf{a_d}$. Finally, if there exist **P**>**0** and $\tau\ge 0$ which satisfy the LMI (B.4).

$$\begin{bmatrix}\boldsymbol{A}^T\boldsymbol{P}+\boldsymbol{PA}+\boldsymbol{D}^T\boldsymbol{D}+\tau\boldsymbol{C}^T\boldsymbol{C}&\boldsymbol{PB}&\boldsymbol{P}\\\boldsymbol{B}^T\boldsymbol{P}&-\tau\boldsymbol{I}&\mathbf{0}\\\boldsymbol{P}&\mathbf{0}&-\gamma^2\boldsymbol{I}\end{bmatrix}\le\mathbf{0}\quad\text{(B.4)}$$

then, the forced system (i.e. $d\neq 0$) is L_2 stable by Theorem 1.

Therefore, by Theorem 3, is a globally attractive equilibrium of the unforced system of (7) (i.e. $d\neq 0$). □

APPENDIX C

- **S-procedure of LMI theory [9]**

Let $F_0,\cdots,F_p$ be quadratic functions of the variable $\xi\in R^n$ such that

$$F_i(\zeta)\equiv\zeta^T\mathbf{T}_i\zeta+2\mathbf{u}_i^T\zeta+v_i\quad i=0,...,p\quad \mathbf{T}_i=\mathbf{T}_i^T.$$

We consider the following condition on $F_0,\cdots,F_p$

$$\begin{aligned}&F_0(\xi)\ge 0\ \text{ for all }\ \xi\ \text{ such that}\\&F_i(\xi)\ge 0,\quad i=0,1,\cdots,p.\end{aligned}\quad\text{(C.1)}$$

Obviously, if there exists $\tau_1\ge 0,\cdots,\tau_p\ge 0$ such that

$$\text{for all }\ \xi,F_0(\zeta)-\sum_{i=1}^{p}\tau_iF_i(\zeta)\ge 0,\quad\text{(C.2)}$$

then (30) holds or equivalently (32) holds.

$$\begin{bmatrix}\mathbf{T_0}&\mathbf{u_0}\\\mathbf{u_0^T}&v_0\end{bmatrix}-\sum_{i=1}^{p}\tau_i\begin{bmatrix}\mathbf{T_i}&\mathbf{u_i}\\\mathbf{u_i^T}&v_i\end{bmatrix}\ge 0\quad\text{(C.3)}$$

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