Discrete-Time Sliding Mode Controller for Linear Time-Varying Systems with Disturbances

Kang-Bak Park

Abstract: In this paper, a discrete-time sliding mode controller for linear time-varying systems with disturbances is proposed. The proposed method guarantees that the system state is globally uniformly ultimately bounded (G.U.U.B.) under the existence of time-varying disturbances.

Keywords: sliding mode control, linear time-varying systems.

I. Introduction

Lots of control methodologies have been proposed to control a lot of kind of plants. One of the methods is to make a model of the plant under consideration and then design a control system for the model to be stable. In the actual cases, however, there exists a modelling error. In addition, it is very difficult to measure the actual parameter of the plant exactly. Thus, the model has the uncertainties in its parameters and/or structure.

It has been known that the Sliding Mode Control has robust and invariant property to parameter uncertainties and external disturbances. The sliding mode control is designed for the system state to be forced to stay on the predetermined sliding surface. When the system is in the sliding mode, the overall system shows the invariance property to parameter variations and external disturbances, and the dynamics of the closed-loop system is determined by the prescribed sliding surface. Almost all of previous works of sliding mode control have been studied in the continuous-time domain [1]-[3].

In the actual systems, however, controllers are implemented in the discrete-time domain since they use microprocessors or computers in general. And it is well known that the control system designed in the continuous-time domain may become unstable after sampling.

Recently, a sliding mode control in the discrete-time domain has attracted the attention [4]-[7]. Generally speaking, lots of previous works have used discretized version of continuous-time design schemes for the systems with no uncertainty or disturbance: reaching condition [4], $s(t)\,\dot{s}(t)<0$, i.e.,

$$|s(k+1)| < |s(k)|,$$

or Lyapunov approach [5], $\dot{V}(t)<0$, where

$$V(t)=\frac{1}{2}s^2(t), \text{ i.e.,}$$

$$V(k+1)-V(k)<0,$$

where $V(k)=\frac{1}{2}s^2(k)$. Under the existence of uncertainties and disturbances, however, the discrete- time sliding mode control does not guarantee the invariant property. Furthermore, it does not assure the asymptotic convergence of the system state, either.

Thus, in this paper, a discrete-time sliding mode control for linear time-varying systems is proposed. The proposed method guarantees that the system state is globally uniformly ultimately bounded (G.U.U.B.) under the existence of time-varying disturbance and uncertainty. It is also shown that the closed-loop system is globally asymptotically stable if the disturbance and uncertainty is time-invariant.

II. Problem formulation

Consider a discrete-time linear time-varying plant of the following form:

$$x(k+1)=A(k)x(k)+Bu(k)+d(k), \tag{1}$$

where $k=0,1,2,\cdots$, $x(\cdot)\in R^n$ is the state vector, $u(\cdot)\in R$ is the scalar input, and $d(\cdot)\in R^n$ is the vector of external disturbances. The index k indicates the k-th sample, $A(\cdot)\in R^{n\times n}$ is the linear time-varying system matrix, B is the input matrix of appropriate dimension, and it's assumed that the matching condition is satisfied. For the vector of disturbance, it is also assumed that $d_i(t)\in L^\infty$, $\dot{d}_i(t)\in L^\infty$, and there exists a constant vector $D\in R^n$ such that

$$|d_i(k+1)-d_i(k)| \lesssim D_i$$

where $d_i(k)$ means $d_i(kT)$, T is a sampling period, and $i=1,2,\cdots,n$.

III. Main results

Let the sliding surface as

$$s(k)=Cx(k), \tag{2}$$

where $C^T\in R^n$ is assumed to be designed such that CB is nonsingular and the following sliding dynamics is globally uniformly asymptotically stable:

$$s(k)\equiv 0. \tag{3}$$

From (1), the nominal system can be written as

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$$x(k+1) = A(k)x(k) + Bu(k), \tag{4}$$

where $k = 0, 1, 2, \cdots$. For the above system, the sliding dynamics (3) can be rewritten as

$$\begin{aligned} s(k+1) &= Cx(k+1) = CA(k)x(k) + CBu(k) \\ &= s(k) = Cx(k) = 0, \end{aligned} \tag{5}$$

where $k = 0, 1, 2, \cdots$. From the above equation, the equivalent control input can be obtained as

$$u_{eq} = -(CB)^{-1}CA(k)x(k). \tag{6}$$

For the nominal system, therefore, the dynamics in the sliding mode can be expressed as

$$\begin{aligned} x(k+1) &= A(k)x(k) + Bu_{eq}(k) \\ &= [(I - B(CB)^{-1}C)A(k)]x(k). \end{aligned} \tag{7}$$

Thus, it's equivalent to say that C is assumed to be chosen such that the above sliding dynamics turns to be stable.

Then, the following theorem can be derived for the closed-loop system.

Theorem 1: For the discrete-time linear time- varying system (1) with the proposed controller (8), it is guaranteed that the system state is globally uniformly ultimately bounded(G.U.U.B.):

$$\begin{aligned} u(k) &= u(k-1) + (CB)^{-1}[(\gamma - 1)s(k) \\ &\quad - C\{A(k)x(k) - A(k-1)x(k-1)\}], \end{aligned} \tag{8}$$

where γ is arbitrarily chosen such that $s(k+1) = \gamma s(k)$ is asymptotically stable, i.e., $|\gamma| < 1$.

Proof: Since the system under consideration is in the discrete-time domain, one can compute the delayed unknown external disturbance, $d(k-1)$.

From the equation of the plant (1), $x(k)$ can be obtained as

$$x(k) = A(k-1)x(k-1) + Bu(k-1) + d(k-1).$$

Thus, the delayed unknown disturbance, $d(k-1)$, is

$$d(k-1) = x(k) - A(k-1)x(k-1) - Bu(k-1). \tag{9}$$

Then, $s(k+1)$ can be computed as follows:

$$\begin{aligned} s(k+1) &= Cx(k+1) \\ &= CA(k)x(k) + CBu(k) + Cd(k) \\ &= CA(k)x(k) + CBu(k) + Cd(k) \\ &\quad - Cd(k-1) + Cd(k-1) \\ &= CA(k)x(k) + CBu(k) + Cd(k) \\ &\quad - Cd(k-1) + Cx(k) \\ &\quad - CA(k-1)x(k-1) - CBu(k-1) \\ &= CBu(k) - CBu(k-1) + CA(k)x(k) \\ &\quad - CA(k-1)x(k-1) + s(k) \\ &\quad + C\{d(k) - d(k-1)\}. \end{aligned} \tag{10}$$

Applying the proposed discrete-time sliding mode control (8) to the above equation, the following equation can be derived:

$$s(k+1) = \gamma s(k) + C\{d(k) - d(k-1)\}. \tag{11}$$

Since $\gamma < 1$, it is clear that

$$\lim_{k\to\infty} |s(k)| \leq \frac{1}{1-|\gamma|} \sum_{i=1}^{n} |C_i| D_i. \tag{12}$$

Thus, the system state $x(k)$ is globally uniformly ultimately bounded(G.U.U.B.) since C is chosen such that $s(k) = Cx(k) \equiv 0$ is asymptotically stable. ■

Remark 1: It is easy to know that the following delayed/past control input, $u(k-1)$, is used to cancel out the unknown disturbance $d(k)$ approximately.

$$\begin{aligned} u(k-1) = {} & (CB)^{-1}[s(k) - CA(k-1)x(k-1) \\ & - Cd(k-1)]. \end{aligned}$$

For the simple second-order systems with canonical form, the following Corollary can be derived for the bound of the system state.

Corollary 1: For the second-order systems of the canonical form, i.e., $x_1(k+1) = x_2(k)$, if the sliding surface is defined as

$$s(k) = Cx(k) = c_1x_1(k) + c_2x_2(k) = c_1x_1(k) + x_2(k),$$

where $|c_1| < 1$, then the ultimate bound of $x_1(k)$ can be found as the following:

$$\lim_{k\to\infty} |x_1(k)| \leq \frac{1}{1-|c_1|}\left(\frac{1}{1-|\gamma|} \sum_{i=1}^{n} |C_i| D_i\right). \tag{13}$$

Proof: Since it is obvious from Eq. (12), the proof is omitted. ■

Corollary 2: From (12), it is easily known that the closed-loop system is globally uniformly asymptotically stable if the disturbances are time- invariant, that is, $d(\cdot)$ is a constant vector, i.e.,

$$D = \mathbf{0}.$$

It implies that there is no need for the disturbance to be zero for the asymptotic stability.

Proof: Since it is clear, the proof is omitted. ■

Remark 2: The proposed method uses the bound of the variation/difference of the disturbance,

$$D_i \geq |d_i(k+1) - d_i(k)|,$$

where $k = 0, 1, 2, \cdots$. It is clear that $\|D\|$ decreases as the sampling frequency increases. Thus, how large the bound of disturbance $\|D\|$ is, the magnitude of the ultimate bound of the state $x(k)$ can be made very small if the disturbance $d(k)$ varies slowly or the sampling period is set very short.

IV. Simulation results

Consider the following discrete-time system matrices which is similar to that of Myszkorowski [8]:

$$A(k) = \begin{bmatrix} 0 & 1 \\ a_{21}(k) & 0.2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad d(k) = \begin{bmatrix} 0 \\ d_2(k) \end{bmatrix},$$

where $a_{21}(k) = 0.05\cos(0.1\pi k) + 0.24$, and $d_2(k) = 0.05\cos(0.1\pi k) + 0.05\sin(0.05\pi k)$.

The gains for the control system are chosen as $C = [0.8 \quad 1]$ and $\gamma = 0.8$.

As can be seen in Figs. 1 and 2, $x_1(k)$ and $s(k)$ are ultimately bounded under the existence of the time-varying uncertainty and disturbance, $d(k)$.

Figure 3 shows the control input profile of the proposed controller.

From (12) and (13), it is expected that the bound of the system state in the steady-state increases when $|\gamma|$

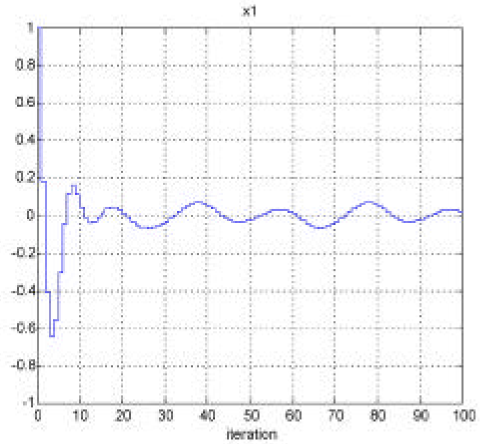

Fig. 1. System state(x1) for γ=0.8.

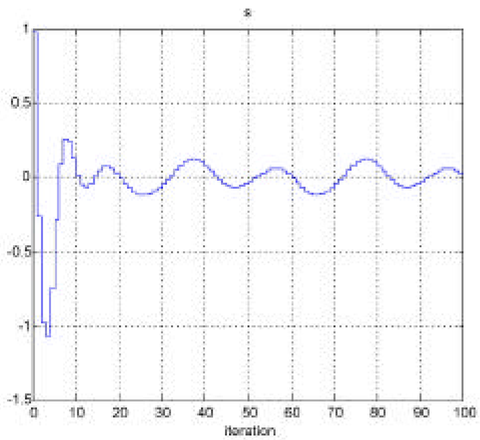

Fig. 2. Sliding surface for γ=0.8.

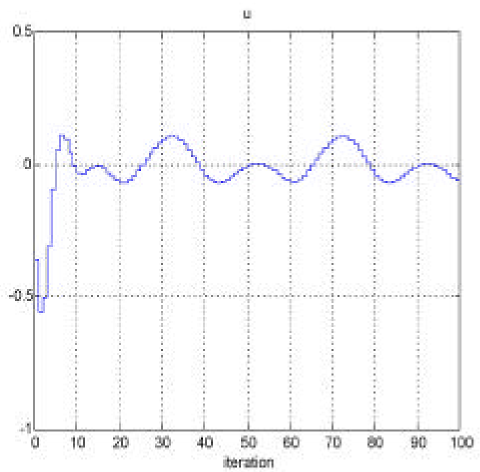

Fig. 3. Control input for γ=0.8.

decreases. Figs. 4 and 5 show the $x_1(k)$ and $s(k)$ when $\gamma = 0.2$. By comparing these figures with Figs. 1 and 2, one can see the effect of the variation of γ, that is, the ultimate bounds are smaller than those of Fig. 1 and 2. And the control signal for this case is shown in Fig. 6.

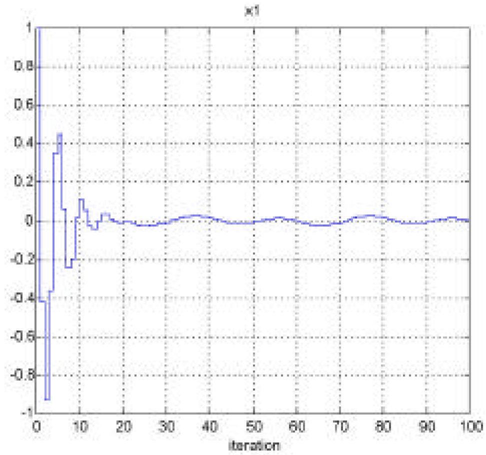

Fig. 4. System state(x1) for γ=0.2.

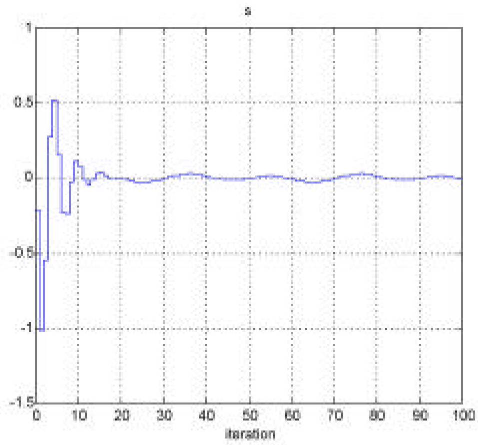

Fig. 5. Sliding surface for γ=0.2.

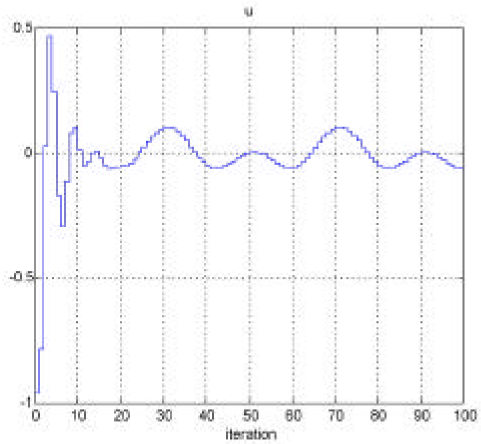

Fig. 6. Control input for γ=0.2.

V. Conclusions

In this paper, a discrete-time sliding mode controller for linear time-varying systems with time-varying disturbances has been presented. It has been shown that the system state is globally uniformly ultimately bounded (G.U.U.B.) under the existence of time- varying disturbance and uncertainty. Moreover, it has been known that although the magnitude of the bound of disturbance and uncertainty is large, the magnitude of the ultimate bound can be set very small by increasing the sampling frequency.

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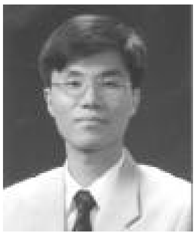

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