Robust Nonlinear H_∞ FIR Filtering for Time-Varying Systems

Hee-Seob Ryu, Won-Kee Son and Oh-Kyu Kwon

Abstract: This paper investigates the robust nonlinear H_∞ filter with FIR(Finite Impulse Response) structure for nonlinear discrete time-varying uncertain systems represented by the state-space model having parameter uncertainty. Firstly, when there is no parameter uncertainty in the system, the discrete-time nominal nonlinear H_∞ FIR filter is derived by using the equivalence relationship between the FIR filter and the recursive filter, which corresponds to the standard nonlinear H_∞ filter. Secondly, when the system has the parameter uncertainty, the robust nonlinear H_∞ FIR filter is proposed for the discrete-time nonlinear uncertain systems.

Keywords: H_∞ FIR filter, nonlinear filtering, robustness, uncertainty, time-varying system

I. Introduction

Over the past several years, the problem of the nonlinear H_∞ filtering has been studied by a number of authors [4][6][12]. There are two approaches commonly used for providing solutions to nonlinear H_∞ control and filtering problems. One is based on the dissipativity theory and the differential game theory. Another is based on the nonlinear version of the classical Bounded Real Lemma developed by Willems [14] and Hill and Moylan [5]. However, the nonlinear H_∞ filters proposed so far are mainly limited to time-invariant systems. Therefore they can not to be applied to general time-varying systems on the infinite horizon since one of two Riccati differential equations required to solve the problem can not be computed on the infinite horizon. To solve this problem, Kwon *et al.* [10][11] have proposed the robust H_∞ FIR filter for general time-varying system. However, their filter is limited to linear systems, and it is not to be directly applied to nonlinear systems.

This paper deals with the issue of the robust nonlinear H_∞ filtering problem for discrete time-varying systems with the parameter uncertainties on the infinite horizon. The basic idea of the current paper is to formulate the robust nonlinear H_∞ filtering problem on the discrete-time moving horizon and to adopt the FIR (Finite Impulse Response) filter structure. FIR filters are widely used in the signal processing area, and they were utilized in the estimation problem as the optimal FIR filters [7][8][9]. Since the optimal FIR filters use the finite observations only over a finite preceding time interval, they can overcome the divergence problem and have the built-in BIBO (Bounded Input/Bounded Output) stability and the robustness to the numerical problems such as coefficient quantization errors and roundoff errors, which are well known properties of the FIR structure in signal processing area. Also note that IIR (Infinite Impulse Response) or recursive filter structure requires the initial conditions on each horizon, which is an impractical assumption, but that FIR filter structure does not requires the initial conditions. The optimal FIR filters are, however, presented so far not in the H_∞ setting but in the minimum variance formulation.

The nonlinear H_∞ filter proposed is to be called hereafter as the *robust nonlinear H_∞ FIR filter* in the sense that it is a nonlinear H_∞ filter with the FIR structure for uncertain systems. It will be shown that the nonlinear H_∞ FIR filter always has a solution if the standard nonlinear H_∞ filter exists on the finite horizon. Therefore, the derivation for H_∞ FIR filter solution for infinite horizon is very simple and trivial. It is noted that the nonlinear filter proposed works on the time-varying nonlinear systems with time-varying parametric uncertainties, and that this point will be one of the main contributions of the current paper.

For the case when there is no parameter uncertainty in the system, we are concerned with designing a nonlinear H_∞ FIR filter such that the induced l_2 operator norm of the mapping from the noise signal to the estimation error is within a specified bound. It is shown that this problem can be solved via one Riccati equation. The design of nonlinear filters which guarantee a prescribed H_∞ performance in the presence of parameter uncertainty are also considered. In this situation, a solution is to be obtained in terms of two Riccati equations.

II. Problem formulation

Consider the uncertain nonlinear time-varying system of the form

$$x_{k+1} = (A + \Delta A_k)x_k + Gg(x_k) + Bw_k \quad (1)$$

$$y_k = (C + \Delta C_k)x_k + Hh(x_k) + Dw_k \quad (2)$$

$$z_k = Lx_k, \quad (3)$$

where $x_k \in \Re^n$ is the state vector with the initial state x_0 unknown, $w_k \in \Re^q$ is a noise signal which belongs to $l_2[0,\infty)$, $y_k \in \Re^m$ is the measurement, $z_k \in \Re^p$ is a linear combination of state variables to be estimated, $g(\cdot):\Re^n \to \Re^{n_g}$ and $h(\cdot):\Re^n \to \Re^{n_h}$ are known nonlinear vector functions and A, B, C, D, G, H and L are known real time-varying matrices of appropriate dimensions that describe the nominal system together with $g(\cdot)$ and $h(\cdot)$. The matrices ΔA_k and ΔC_k represent time-varying parameter uncertainties in A and C, respectively. These uncertainties are assumed to be of the following structure

$$\begin{bmatrix} \Delta A_k \\ \Delta C_k \end{bmatrix} = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} F_k E, \quad (4)$$

where F_k is an unknown real time-varying matrix satisfying

$$F_k^T F_k \le I, k = 0,1,2,\cdots \quad (5)$$

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and H_1, H_2 and E are known real constant matrices of appropriate dimensions that specify how the elements of the nominal matrices A and C are affected by the uncertain parameters in F_k.

Assumption 1:

(a) $[D \quad H_2 \quad H]$ *is of the full row rank*;

(b) $DB^T = 0$;

(c) $g(0) = 0$;

(d) *There exist known constant matrices* V_1 and V_2 *such that for any* x_1 *and* $x_2 \in \Re^n$,

$$\|g(x_1) - g(x_2)\| \le \|V_1(x_1 - x_2)\|$$
$$\|h(x_1) - h(x_2)\| \le \|V_2(x_1 - x_2)\|.$$

Assumption 1(a) and 1(b) means that the robust H_∞ FIR filtering problem is 'nonsingular'. Observe that if the parameter uncertainty and the nonlinearity in the output matrix disappear, i.e. $H_2 = 0$ and $H = 0$, Assumption 1(a) reduces to $DD^T > 0$, which is a standard assumption in the H_∞ FIR filtering problem for the nominal system. Assumption 1(c) means that the initial condition of the nonlinearity function in the state matrix is zero.

Observe that discrete-time nonlinear models of the form (1)-(2) can be used to represent many important physical systems. The parameter uncertainty in the linear terms can be regarded as the variation of the operating points of the nonlinear system. We also note that the parameter uncertainty structure of (4) has been widely used in the problems of robust control and robust filtering and can capture the uncertainty in a number of practical situations.

In this section we are concerned with designing a nonlinear causal filter $\Im$ with FIR structure for estimating z_{ek} with a guaranteed performance in a H_∞ sense, using the measurements $Y_{k-1} = \{y_j, j = 0,1,\cdots k-1\}$ and where no *a priori* estimate of the initial state of (1) is assumed. Letting z_{ek} denote the estimate of z_k, the filter is required to guarantee a uniformly small estimation error $z_k - z_{ek}$, for any $w \in l_2[0,\infty)$ and $x_0 \in \Re^n$. Then the *robust nonlinear* H_∞ *FIR filtering problem* is formulated as follows:

Given the system (1)-(3) and a prescribed level of noise attenuation $g > 0$ *on each horizon* $[k-N, k]$, *find a causal filter* $\Im$ *such that the filtering error dynamics is globally uniformly asymptotically stable and* $\|z - z_e\|^2 < g\{\|w\|_{N2}^2 + x_0^T R x_0\}$ *for any non-zero* $(x_0, w) \in \Re^n \oplus l_2[0,\infty)$ *and for all uncertainties satisfying (4)-(5), where* $R = R^T > 0$ *is a given weighting matrix for* x_0 *and '* $\oplus$ *' means the direct sum of the vector subspaces. Here,* $\|e\|$ *denotes* $e^T e$ *on the infinite horizon and* $\|\cdot\|_{N2}$ *denotes the usual* l_2*-norm on the moving horizon* $[k-N, k]$.

Provided that there is no parameter uncertainty in the system, *i.e.*, $\Delta A_k = 0$ and $\Delta C_k = 0$ for all k in the above formulation, the problem reduces to the *nominal nonlinear* H_∞ *FIR filtering problem*, which corresponds to the nonlinear H_∞ filtering problem.

Note that the performance index in the above problem statements is a worst-case performance measure and can be viewed as a generalization of the standard H_∞ performance measure to deal with unknown initial state. The weighting matrix, R, is a measure of the uncertainty in x_0 relative to the uncertainty in w. A 'large' value of R indicates that the initial state is likely to be very close to zero.

In the current paper, the FIR filter is defined as follows

$$\hat{x}(k+1|k;N) = \sum_{i=k-N}^{i} T(k,i;N)y(i)$$
$$\hat{z}(k+1|k;N) = L(k+1)\hat{x}(k+1|k;N),$$

where $T(k,\cdot;N)$ is the finite impulse response with the finite duration N. This FIR filter is a kind of the one-step-ahead predictor since it estimates the state or the output at the time point $k+1$ based on the observation on $[k-N, k]$. The H_∞ FIR filter is obtained by constructing its impulse response from that of the H_∞ filter on the finite moving horizon $[k-N, k]$.

We end this section by recalling a version of the bounded real lemma for linear discrete time-varying systems, which will be used in the derivation of a solution to the above filtering problems.

Consider the following linear time-varying system

$$x_{k+1} = A_k x_k + B_k w_k \tag{6}$$

$$z_k = C_k x_k, \tag{7}$$

where $x_k \in \Re^n$ is the state vector with the initial state x_0 being unknown, $w_k \in \Re^q$ is the input which belongs to $l_2[0,\infty)$, $z_k \in \Re^p$ is the measurement, and A_k, B_k and C_k are known bounded real time-varying matrices. Also, we define the following worst-case performance measure for the system (6)-(7):

$$J(z, w, x_0, R) = \sup_{(x_0, w) \neq 0} \left\{ \left[\frac{\|z\|^2}{\|w\|_{N2}^2 + x_0^T R x_0} \right]^{1/2} \right\},$$

where $R = R^T > 0$ is a given weighting matrix for the initial state and $0 \neq (x_0, w) \in \Re^n \oplus l_2[0,\infty)$. Then, we have the following result.

Lemma 1 [16]: Consider the system (6)-(7) and let $g > 0$ be a given scalar. Then, the following statements are equivalent:

(a) The system (6) is exponentially stable and $J < g$;

(b) There exists a bounded time-varying matrix $Q_k = Q_k^T \ge 0$, $\forall k \ge 0$, satisfying $I - g^{-2} C_k Q_k C_k^T > 0$, $\forall k \ge 0$, and such that

$$A_k Q_k A_k^T - Q_{k+1} + g^{-2} A_k Q_k C_k^T (I - g^{-2} C_k Q_k C_k^T)^{-1}$$
$$\cdot C_k Q_k A_k^T + B_k B_k^T = 0, \qquad Q_0 = R^{-1},$$

and the system

$$x_{k+1} = [A_k + g^{-2} A_k Q_k C_k^T (I - g^{-2} C_k Q_k C_k^T)^{-1} C_k] x_k$$

is exponentially stable;

(c) There exists a bounded time-varying matrix $P_k = P_k^T > 0$, $\forall k \ge 0$, satisfying $I - g^{-2} B_k^T P_{k+1} B_k > 0$, $\forall k \ge 0$, and such that

$$A_k^T P_{k+1} A_k - P_k + g^{-2} A_k^T P_{k+1} B_k (I - g^{-2} B_k^T P_{k+1} B_k)^{-1}$$

$$\cdot B_k^T P_{k+1} A_k + C_k^T C_k < 0, \quad P_0 < g^2 R.$$

Observe that when the initial state of (6) is zero, the performance index $J(z,w,x_0,R)$ becomes the usual H_∞ performance measure, namely

$$J(z,w) = \sup_{0 \neq (x_0,w) \in l_2[0,\infty)} \left\{ \frac{\| z \|}{\| w \|_{N2}} \right\}.$$

The index of performance $J(z,w)$ can be viewed as the limit of $J(z,w,x_0,R)$ as the smallest eigenvalue of R approaches infinity. In this case it happens that $Q_0 = 0$ in the statement (b) of Lemma 1 while the requirement $P_0 < g^2 R$ in the statement (c) will become superfluous.

Firstly, in the current paper, the *nonlinear H_∞ FIR filtering problem* will be solved, and then the *robust nonlinear H_∞ FIR filtering problem* is to be dealt with. It is noted that the problem does not need the assumption of stabilizability or detectability of the system since it is formulated on the finite moving horizon.

III. Nonlinear H_∞ FIR filters

In the sequel we shall provide a solution to both the problems of nominal and robust H_∞ filtering with FIR structure using a Riccati equation approach. We first present a performance analysis result for the system (1)-(3).

Theorem 1: Consider the system (1)-(3) satisfying Assumption 1. Given a scalar $g > 0$ and an initial state weighting matrix $R = R^T > 0$ then, the system (1) is globally uniformly asymptotically stable and $\| z \| < g\{\| w \|_{N2}^2 + x_0^T R x_0\}^{1/2}$ for any nonzero $(x_0, w) \in \Re^n \oplus l_2[0,\infty)$ and for all ΔA_k satisfying (4)-(5) if there exist a scalar $e > 0$ and a bounded time-varying matrix $Q_k = Q_k^T \geq 0$, $\forall k \geq 0$, satisfying $I - g^{-2} B_1^T Q_{k+1} B_1 > 0$, $\forall k \geq 0$, and such that

$$A^T Q_{k+1} A - Q_k + g^{-2} A^T Q_{k+1} B_1 (I - g^{-2} B_1^T Q_{k+1} B_1)^{-1} B_1^T Q_{k+1} A + L^T L + e^2 E^T E + V_1^T V_1 < 0, \quad Q_0 < g^2 R,$$

where

$$B_1 = \begin{bmatrix} B & \frac{g}{e} H_1 & gG \end{bmatrix}. \quad (8)$$

Proof: Define a Lyapunov function candidate $V(x_k) = x_k^T Q_k x_k$. Since $d_1 I < Q_k < d_2 I$, $V(x_k)$ is positive definite and decrescent. It can be easily shown that $\Delta V(x_k) = V(x_{k+1}) - V(x_k) \leq -d_1 x_k^T$ along the trajectory of (1). Hence, $V(x_k)$ is a Lyapunov function and the system (1) is globally uniformly asymptotically stable. Remaining parts of the proof can be easily established similarly to the proof of Theorem 4.2 in [15]. ■

In the case when there is no parameter uncertainty in (1), Theorem 1 reduces to the following corollary.

Corollary 1: Consider the system (1)-(3) with $\Delta A_k \equiv 0$ and satisfying Assumption 1. Given a scalar $g > 0$ and an initial state weighting matrix $R = R^T > 0$ then, the system of (1) is globally uniformly asymptotically stable and $\| z \| < g\{\| w \|_{N2}^2 + x_0^T R x_0\}^{1/2}$ for any non-zero $(x_0, w) \in \Re^n \oplus l_2[0,\infty)$ if there exist a scalar $e > 0$ and a bounded time-varying matrix $Q_k = Q_k^T \geq 0$, $\forall k \geq 0$, satisfying $I - g^{-2} B_1^T Q_{k+1} B_1 > 0$, $\forall k \geq 0$, and such that

$$A^T Q_{k+1} A - Q_k + g^{-2} A^T Q_{k+1} \overline{B}_1 (I - g^{-2} \overline{B}_1^T Q_{k+1} \overline{B}_1)^{-1} \overline{B}_1^T Q_{k+1} A + L^T L + V_1^T V_1 < 0, \quad Q_0 < g^2 R,$$

where $\overline{B}_1 = [B \quad gG]$.

Note that when the initial state of the system (1) is known to be zero, the time-varying matrix Q_k in Theorem 1 and Corollary 1 may be replaced by a constant matrix $Q = Q^T \geq 0$. Furthermore, the condition $Q < g^2 R$ will no longer be required as an initial state which is certain to be zero corresponds to choosing a `very large' value of R.

We now present a solution to the nominal nonlinear H_∞ FIR filtering problem for the system (1)-(3).

Theorem 2: Consider the system (1)-(3) with $\Delta A_k \equiv 0$ and $\Delta C_k \equiv 0$, and satisfying Assumption 1. Given a scalar $g > 0$ and an initial state weighting matrix $R = R^T > 0$, the nominal nonlinear H_∞ FIR filtering problem is solvable if there exists a bounded time-varying matrix $S_k = S_k^T > 0$, $\forall k \geq 0$, satisfying $I - g^{-2} \tilde{L} S_k \tilde{L}^T > 0$, $\forall k \geq 0$,

$$S_{k+1} = A S_k A^T - (A S_k \tilde{C}^T + \tilde{B}\tilde{D}_1^T)(\tilde{C} S_k \tilde{C}^T + \tilde{R})^{-1} \cdot (\tilde{C} S_k A^T + \tilde{D}_1 \tilde{B}^T) + \tilde{B}\tilde{B}^T, \quad S_0 = R^{-1}, \quad (9)$$

and the system

$$\begin{aligned} r_{k+1} &= A_{1k} r_k \\ &= [A - (A S_k \tilde{C}_1^T + \tilde{B}\tilde{D}_1^T)(\tilde{R} + \tilde{C}_1 S_k \tilde{C}^T)^{-1} \tilde{C}] r_k \end{aligned} \quad (10)$$

is exponentially stable, where

$$\tilde{L}^T \tilde{L} = L^T L + V^T V, \quad V = \begin{bmatrix} V_1^T & V_2^T \end{bmatrix}^T,$$

$$\tilde{B} = \begin{bmatrix} B & gG & 0 \end{bmatrix}, \quad \tilde{D} = \begin{bmatrix} D & 0 & gH \end{bmatrix},$$

$$\tilde{C}_1 = \begin{bmatrix} C \\ g^{-1} \tilde{L} \end{bmatrix}, \quad \tilde{D}_1 = \begin{bmatrix} \tilde{D} \\ 0 \end{bmatrix}, \quad \tilde{R} = \begin{bmatrix} \tilde{D}\tilde{D}^T & 0 \\ 0 & -I \end{bmatrix}.$$

Moreover, if the above conditions hold, a suitable nonlinear filter is given by

$$x_{e(k+1)} = A x_{ek} + G g(x_{ek}) + K_{ek}[y_k - C x_{ek} - H h(x_{ek})] \quad (11)$$

$$z_{ek} = L x_{ek}, \quad (12)$$

where

$$K_{ek} = (A \hat{S}_k C^T + \tilde{B}\tilde{D}^T)(C \hat{S}_k C^T + \tilde{D}\tilde{D}^T)^{-1} \quad (13)$$

$$\hat{S}_k = S_k + g^{-2} S_k \tilde{L}^T (I - g^{-2} \tilde{L} S_k \tilde{L}^T)^{-1} \tilde{L} S_k. \quad (14)$$

Proof: Firstly, note that the condition $I - g^{-2} \tilde{L} S_k \tilde{L}^T$, $\forall k \geq 0$, together with Assumption 1 guarantees the nonsingularity of the matrix $\tilde{R} + \tilde{C} S_k \tilde{C}^T$, $\forall k \geq 0$. Letting $\tilde{x}_k = x_k - x_{ek}$ and $e_k = z_k - z_{ek}$, it follows from (1)-(3) (setting $\Delta A_k \equiv 0$ and $\Delta C_k \equiv 0$) and (11)-(12) that

$$\begin{aligned} \tilde{x}_{k+1} = (A - K_{ek} C)\tilde{x}_k + (G_1 - K_{ek} H_1) x(x_k, x_{ek}) \\ + (B - K_{ek} D) w_k \end{aligned} \quad (15)$$

$$e_k = L \tilde{x}_k, \quad (16)$$

where

$$x(x_k, x_{ek}) = \begin{bmatrix} g(x_k) - g(x_{ek}) \\ h(x_k) - h(x_{ek}) \end{bmatrix}$$

and $G = [G \quad 0]$, $H_1 = [0 \quad H]$.
Note that by Assumption 1, $\|x(x_k, x_{ek})\| \le \|V\tilde{x}_k\|$.

It can be shown from (9) that $Q_k = g^{-2} S_k$ is such that $I - \tilde{L} Q_k \tilde{L}^T > 0, \quad \forall k \ge 0$, and satisfies

$$(A - K_{ek}C) Q_k (A - K_{ek}C)^T - Q_{k+1} + (A - K_{ek}C) Q_k \tilde{L}^T \cdot (I - \tilde{L} Q_k \tilde{L}^T)^{-1} \tilde{L} Q_k (A - K_{ek}C)^T + g^{-2} \tilde{B}_1 \tilde{B}_1^T = 0,$$
$$Q_0 = g^{-2} R^{-1}, \tag{17}$$

where $\tilde{B}_1 = [(B - K_{ek}D) \quad g(G_1 - K_{ek}H_1)]$.
Also, it is easy to verify that the state matrix A_{1k} of the system of (10) can be rewritten as

$$A_{1k} = (A - K_{ek}C)[I + Q_k \tilde{L}^T (I - \tilde{L} Q_k \tilde{L}^T)^{-1} \tilde{L}].$$

Since the system of (10) is exponentially stable, in view of Lemma 1 and Corollary 1, (17) implies that the estimation error dynamics of (15)-(16) is globally uniformly asymptotically stable and $\|e\| < g\{\|w\|_{N2}^2 + x_0^T R x_0\}^{1/2}$ for any non-zero $(x_0, w) \in \Re^n \oplus l_2[0, \infty)$. ■

When the initial state of the system of (1) is known to be zero, and a stationary filter design is concerned, Theorem 2 can be specialized as follows.

Theorem 3: Consider the system (1)-(3) with $x_0 = 0$, $\Delta A_k \equiv 0$, $\Delta C_k \equiv 0$, and satisfying Assumption 1. Given a scalar $g > 0$ and an initial state weighting matrix $R = R^T > 0$, the nominal nonlinear H_∞ FIR filtering problem is solvable if there exists a stabilizing solution $S = S^T > 0$, to the algebraic Riccati equation

$$S = ASA^T - (AS\tilde{C}^T + \tilde{B}\tilde{D}_1^T)(\tilde{C}S\tilde{C}^T + \tilde{R})^{-1} \cdot (\tilde{C}SA^T + \tilde{D}_1\tilde{B}^T) + \tilde{B}\tilde{B}^T \quad . \tag{18}$$

such that $I - g^{-2} \tilde{L} S \tilde{L}^T > 0$. Moreover, if the above conditions hold, a suitable nonlinear filter is given by (11)-(12), where the filter gain of (13) is constant. ■

It should be pointed out that in Theorems 2 and 3 no stability requirement is imposed on the system (1). We also observe that, when there are no nonlinear terms in the system (1)-(2), i.e. $g(\cdot) \equiv 0$ and $h(\cdot) \equiv 0$, the result of Theorem 3 will reduce to the H_∞ FIR linear filter.

IV. Robust nonlinear H_∞ FIR filters

Next, we solve the robust nonlinear H_∞ FIR filtering problem. To this end, we shall make a further assumption on the system (1).

Assumption 2: The nominal state matrix A is nonsingular.

Theorem 4: Consider the uncertain system (1)-(3) satisfying (4)-(5) and Assumptions 1-2. Let $n > 0$ be an arbitrary small scalar. Given a scalar $g > 0$ and an initial state weighting matrix $R = R^T > 0$, the robust H_∞ FIR filtering problem is solvable if for some scalar $e > 0$, the following conditions hold:

(a) There exists a stabilizing solution $P = P^T > 0$ to the algebraic Riccati equation:

$$A^T PA - P + g^{-2} A^T PB_1 (I - g^{-2} B_1^T PB_1)^{-1} B_1^T PA + E_1^T E_1 + nI = 0 \tag{19}$$

such that $I - g^{-2} B_1^T PB_1 > 0$, and $P < g^2 R$, where B_1 is as in (17) and

$$E_1^T E_1 = e^2 E^T E + V_1^T V_1 ; \tag{20}$$

(b) There exists a bounded time-varying matrix $S_k = S_k^T \ge 0$, $\forall k \ge 0$, satisfying $I - g^{-2} \hat{L} S_k \hat{L}^T > 0, \forall k \ge 0$, and such that

$$S_{k+1} = \hat{A} S_k \hat{A}^T - (\hat{A} S_k \hat{C}_1^T + \hat{B}\hat{D}_1^T)(\hat{C}_1 S_k \hat{C}_1^T + \hat{R})^{-1} \cdot (\hat{C}_1 S_k \hat{A}^T + \hat{D}_1 \hat{B}^T) + \hat{B}\hat{B}^T, \quad S_0 = (R - g^{-2} P)^{-1}, \tag{21}$$

and the system

$$r_{k+1} = A_{2k} r_k = [\hat{A} - (\hat{A} S_k \hat{C}_1^T + \hat{B}\hat{D}_1^T)(\hat{C}_1 S_k \hat{C}_1^T + \hat{R})^{-1} \hat{C}_1] r_k \tag{22}$$

is exponentially stable, where

$$\hat{L}^T \hat{L} = L^T L + V^T V, \quad V = \begin{bmatrix} V_1^T & V_2^T \end{bmatrix}^T, \tag{23}$$

$$\hat{C}_1 = \begin{bmatrix} \hat{C} \\ g^{-1}\hat{L} \end{bmatrix}, \quad \hat{D}_1 = \begin{bmatrix} \hat{D} \\ g^{-1}\hat{L} \end{bmatrix}, \quad \hat{R} = \begin{bmatrix} \hat{D}\hat{D}^T & 0 \\ 0 & -I \end{bmatrix}, \tag{24}$$

$$\hat{A} = A + \Delta A_e = A + g^{-2} \overline{B}\overline{B}^T (P^{-1} - g^{-2} B_1 B_1^T)^{-1} A, \tag{25}$$

$$\hat{C} = C + \Delta C_e = C + g^{-2} \overline{D}\overline{B}^T (P^{-1} - g^{-2} B_1 B_1^T)^{-1} A, \tag{26}$$

$$\hat{B} = \begin{bmatrix} \overline{B}M & gG & 0 \end{bmatrix}, \tag{27}$$

$$\hat{D} = \begin{bmatrix} \overline{D}M & 0 & gH \end{bmatrix}, \tag{28}$$

$$\overline{B} = \begin{bmatrix} B & \frac{g}{e} H_1 \end{bmatrix}, \quad \overline{D} = \begin{bmatrix} D & \frac{g}{e} H_2 \end{bmatrix}, \tag{29}$$

$$M = [I + g^{-2} \overline{B}^T (P^{-1} - g^{-2} B_1 B_1^T) \overline{B}]^{-1/2} . \tag{30}$$

Moreover, if conditions (a) and (b) are satisfied, a suitable nonlinear filter is given by

$$x_{e(k+1)} = \hat{A} x_{ek} + Gg(x_{ek}) + K_{ek}[y_k - \hat{C} x_{ek} - Hh(x_{ek})] \tag{31}$$

$$z_{ek} = Lx_{ek}, \tag{32}$$

where

$$K_{ek} = (\hat{A}\hat{S}_k \hat{C}^T + \hat{B}\hat{D}^T)(\hat{C}\hat{S}_k \hat{C}^T + \hat{D}\hat{D}^T)^{-1} \tag{33}$$

$$\hat{S}_k = S_k + g^{-2} S_k \hat{L}^T (I - g^{-2} \hat{L} S_k \hat{L}^T)^{-1} \hat{L} S_k . \tag{34}$$

Proof: First, note that since $P > 0$ and $I - g^{-2} B_1^T PB_1 > 0$, it follows that the matrix $P^{-1} - g^{-2} B_1 B_1^T$ is positive definite. Hence, the coefficient matrices of (21) are well defined. Moreover, the condition $I - g^{-2} \hat{M} S_k \hat{M}^T > 0, \quad \forall k \ge 0$, together with Assumption 1(a) guarantee the non-singularity of $\hat{C}_1 S_k \hat{C}_1^T + \hat{R}$ for all $\forall k \ge 0$.

Next, we rewrite the filter (31)-(32) in the following form:

$$x_{e(k+1)} = (A+\Delta A_e)x_{ek} + Gg(x_{ek}) + K_{ek}[y_k - (C+\Delta C_e)x_{ek} - Hh(x_{ek})] \quad (35)$$

$$z_{ek} = Mx_{ek}, \quad (36)$$

where ΔA_e and ΔC_e are given in (25)-(26), respectively. We note that ΔA_e and ΔC_e reflect the effect of the parameter uncertainties ΔA_k and ΔC_k on the filter structure. When ΔA_k and ΔC_k in the system (1)-(3) disappear, ΔA_e and ΔC_e in (35) will naturally be set to zero.

Defining $\tilde{x} = x - x_e$, from (1)-(3) and (35)-(36) we obtain that

$$h_{k+1} = (A_c + H_c F_k E_c)h_k + G_c g_c(x_k, x_{ek}) + B_c w_k \quad (37)$$

$$e_k = C_c h_k,\ h = \begin{bmatrix} x^T & \tilde{x}^T \end{bmatrix}^T, \quad (38)$$

where $e = z - z_e$,

$$A_c = \begin{bmatrix} A & 0 \\ -(\Delta A_e - K_{ek}\Delta C_e) & A + \Delta A_e - K_{ek}(C+\Delta C_e) \end{bmatrix}$$

$$B_c = \begin{bmatrix} B \\ B - K_{ek}D \end{bmatrix},\quad H_c = \begin{bmatrix} L_1 \\ L_1 - K_{ek}L_2 \end{bmatrix}$$

$$C_c = \begin{bmatrix} 0 & M \end{bmatrix},\quad E_c = \begin{bmatrix} E & 0 \end{bmatrix},$$

$$G_c = \begin{bmatrix} G & 0 & 0 \\ 0 & G & -K_{ek}H \end{bmatrix},\quad g_c(x_k, x_{ek}) = \begin{bmatrix} g(x_k) \\ g(x_k) - g(x_{ek}) \\ h(x_k) - h(x_{ek}) \end{bmatrix}.$$

Note that by Assumption 1 we have that

$$\| g_c(x_k, x_{ek}) \| \le \| \hat{V} h_k \|,$$

where $\hat{V} = diag(V_1, V)$ and V is as in (23).

Next, it can be shown by using standard, but tedious, matrix manipulations that

$$X_k = \begin{bmatrix} P^{-1} & 0 \\ 0 & g^{-2}S_k \end{bmatrix}, \quad (39)$$

where P and S_k are the required solutions of (19) and (21) satisfies

$$A_c S_k A_c^T - X_{k+1} + A_c X_k \hat{C}_c^T (I - \hat{C}_c X_k \hat{C}_c^T)^{-1} \cdot \hat{C}_c X_k A_c^T + \hat{B}_c \hat{B}_c = 0; \quad X_0 = \tilde{R}^{-1}, \quad (40)$$

where $\tilde{R} = diag(P, g^2 R - P) > 0$.

We now show that X_k given in (39) is such that the time-varying system

$$r_{k+1} = \hat{A}_c r_k = [A_c + A_c X_k \hat{C}_c^T (I - \hat{C}_c X_k \hat{C}_c^T)^{-1} \hat{C}_c] r_k \quad (41)$$

is exponentially stable. It can be shown that

$$\hat{A}_c = \begin{bmatrix} \overline{A} & 0 \\ * & A_{2k} \end{bmatrix},$$

where A_{2k} is as in (22), '$*$' denotes entries which are bounded but irrelevant, and

$$\overline{A} = A + g^{-2} B_1 (I - g^{-2} B_1^T P B_1)^{-1} B_1^T PA.$$

Note that as P is the stabilizing solution of (19), $\overline{A}$ is Schur stable. Moreover, since the system (22) is exponentially stable, it follows that the system (41) is exponentially stable as well. Hence, X_k is the stabilizing solution of (40).

By Lemma 1, this implies that there exist a scalar $d_1 > 0$ and a bounded time-varying matrix $Y_k = Y_k^T > 0,\ \forall k \ge 0$, satisfying $I - \hat{B}_c^T Y_{k+1} \hat{B}_c > 0,\ \forall k \ge 0$, and such that

$$A_c^T Y_{k+1} A_c - Y_k + A_c^T Y_{k+1} \hat{B}_c (I - \hat{B}_c^T Y_{k+1} \hat{B}_c)^{-1} \hat{B}_c^T Y_{k+1} A_c + \hat{C}_c^T \hat{C}_c < 0; \quad Y_0 < \tilde{R}.$$

Now, taking into account that

$$\hat{C}_c^T \hat{C}_c = C_c^T C_c + e^2 E_c^T E_c + \hat{V}^T \hat{V} + \begin{bmatrix} nI & 0 \\ 0 & 0 \end{bmatrix},$$

we obtain that Y_k satisfies the following inequality

$$A_c^T Y_{k+1} A_c - Y_k + A_c^T Y_{k+1} \hat{B}_c (I - \hat{B}_c^T Y_{k+1} \hat{B}_c)^{-1} \hat{B}_c^T Y_{k+1} A_c + C_c^T C_c + e^2 E_c^T E_c + \hat{V}^T \hat{V} < 0; \quad Y_0 < \tilde{R}.$$

Also, note that $h_0^T \tilde{R} h_0 = g^2 x_0^T R x_0$. Finally, in view of the definition of $\hat{B}_c$, it follows from Theorem 1 that the estimation error system (37)-(38) is globally uniformly asymptotically stable and

$$\| e \| < g \{ \| w \|_{N2}^2 + x_0^T R x_0 \}^{1/2} \quad (42)$$

for all non-zero $(x_0, w) \in \Re^n \oplus l_2[0, \infty)$ and for all admissible uncertainties. ■

The arbitrary small scalar $n > 0$ is introduced in Theorem 4 to guarantee that the stabilizing solution of (19) is positive definite. In the case when $E_1^T E > 0$, or the pair (A, E_1) has no unobservable modes in the closed unit disk, n can be set to zero.

We observe that the existence of a matrix P satisfying condition (a) of Theorem 4 will guarantee the global uniform asymptotic stability of the uncertain system (1) for all uncertainties satisfying (4)-(5). Note that due to the existence of parameter uncertainty in (1), the requirement of global asymptotic stability of (1) is needed in order to ensure the boundedness of the estimation error dynamics for all admissible uncertainties.

It should be noted that the result of Theorem 4 does not recover that of Theorem 2 when the uncertainties ΔA_k and ΔC_k disappear. The reason for this is because when parameter uncertainty exists an asymptotic stability requirement has to be imposed on the system of (1), which gives rise to (19) of Theorem 4.

V. Example

To demonstrate the use of the above theory we consider the robust nonlinear H_∞ FIR filter for a simple second-order problem. We show the advantage of the proposed technique by comparing its results with the corresponding results of the H_∞ nonlinear estimator of Shaked and Berman [13] and the extended Kalman filter (EKF), which has been widely used in the past in estimation of nonlinear systems.

Consider the time-invariant process with a saturating non-

nonlinearity in the system dynamics

$$\begin{bmatrix} x_{1_{i+1}} \\ x_{2_{i+1}} \end{bmatrix} = \begin{bmatrix} mx_{1_i} \\ \tan^{-1}(hx_{1_i} + lx_{2_i}) \end{bmatrix} + \begin{bmatrix} 0.4 \\ 0.4 \end{bmatrix} F(x_i)E(x_i) + \begin{bmatrix} 0.5 \\ 1.5 \end{bmatrix} w_i, \quad (43)$$

where $m = 0.91$, $h = -0.07$, $l = 0.1$, $E(x_i) = [0 \quad 1]x_i$ and $|F(x_i)| \le 1$, $\forall i \in [0, N]$ and $\forall x_i \in \Re^2$.

We assume here that the measurement is described by

$$y_i = \cos(2x_{2_i}) + 3x_{2_i} + 2F(x_i)E(x_i) + 0.01w_i . \quad (44)$$

We consider the time interval $N = 500$, and we are looking for an estimate of Lx_i, where $L = [1 \quad 0]$. We also assumed that $\{w_i\}$ are uncorrelated standard gaussian white noise processes for the comparison of simulation results between robust nonlinear H_∞ FIR filter and EKF.

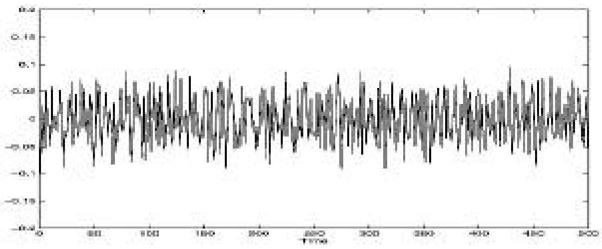
Fig. 1. Estimation error of the robust H_∞ FIR filter.

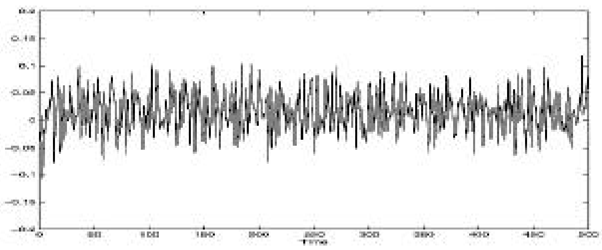
Fig. 2. Estimation error of the robust H_∞ filter.

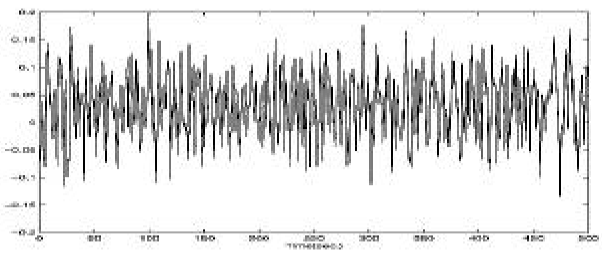
Fig. 3. Estimation error of the extended Kalman filter.

We have simulated the above three estimators for the worst values of the uncertainty F, namely for each estimator we describe the estimators. Fig. 1, 2 and 3 show the estimation error that have been obtained for the three estimators, where $F = 1$ for the robust nonlinear H_∞ FIR filter, the robust nonlinear H_∞ filter of Shaked and Berman [13] and the EKF and g for the robust nonlinear H_∞ FIR filter and robust nonlinear H_∞ filter are 6.3 and 1.1, respectively. Note that estimation error covariances of the proposed nonlinear H_∞ FIR filter, the robust nonlinear H_∞ filter and the extended Kalman filter are 1.5466e-003, 1.7774e-003 and 4.0684e-003, and that the estimation error means of the proposed nonlinear H_∞ FIR filter, the robust nonlinear H_∞ filter and the extended Kalman filter are -2.7307e-004, 1.5995e-002 and 1.4377e-002, respectively. These result exemplify that the estimation performance of the robust H_∞ FIR filter is better than those obtained by the robust nonlinear H_∞ filter and by the Extended Kalman filter.

VI. Conclusions

In this paper the robust nonlinear H_∞ FIR filter has been proposed for nonlinear discrete time-varying systems with parameter uncertainty. Firstly, the discrete-time H_∞ FIR filter is obtained for the nonlinear system without the parametric uncertainty. Secondly, the robust nonlinear H_∞ FIR filter for the uncertain discrete-time nonlinear systems is derived in the modified system model. This paper is an extension of previous works by Kwon *et al.* [10][11] to nonlinear system, which treat linear time-varying systems with parameter uncertainty.

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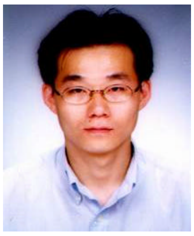

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